

THE STRATIFIED SPACES OF REAL POLYNOMIALS & TRAJECTORY SPACES OF TRAVERSING FLOWS

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ABSTRACT. This paper is the third in a series that researches the Morse Theory, gradient flows, concavity and complexity on smooth compact manifolds with boundary. Employing the local analytic models from [K2], for *transversally generic flows* on $(n+1)$ -manifolds X , we embark on a detailed and somewhat tedious study of universal combinatorics of their tangency patterns with respect to the boundary ∂X . This combinatorics is captured by a universal poset $\Omega_{\langle n \rangle}^\bullet$ which depends only on the dimension of X . It is intimately linked with the combinatorial patterns of real divisors of real polynomials in one variable of degrees which do not exceed $2(n+1)$. Such patterns are elements of another natural poset $\Omega_{\langle 2n+2 \rangle}$ that describes the ways in which the real roots merge, divide, appear, and disappear under deformations of real polynomials. The space of real degree d polynomials \mathcal{P}^d is stratified so that its pure strata are cells, labelled by the elements of the poset $\Omega_{\langle d \rangle}$. This cellular structure in \mathcal{P}^d is interesting on its own right (see Theorem 4.1 and Theorem 4.2). Moreover, it helps to understand the *localized* structure of the trajectory spaces $\mathcal{T}(v)$ for transversally generic fields v , the main subject of Theorem 5.2 and Theorem 5.3.

1. BASICS FACTS ABOUT BOUNDARY GENERIC AND TRAVERSALLY GENERIC VECTOR FIELDS

For the reader convenience, we start with a short review of few key definitions and lemmas from [K1] and [K2].

Let v be a vector field on a smooth compact $(n+1)$ -manifold X with boundary ∂X . To achieve some uniformity in our notations, let $\partial_0 X := X$ and $\partial_1 X := \partial X$.

The vector field v gives rise to a partition $\partial_1^+ X \cup \partial_1^- X$ of the boundary $\partial_1 X$ into two sets: the locus $\partial_1^+ X$, where the field is directed inward of X , and $\partial_1^- X$, where it is directed outwards. We assume that v , viewed as a section of the quotient line bundle $T(X)/T(\partial X)$ over ∂X , is transversal to its zero section. This assumption implies that both sets $\partial_1^\pm X$ are compact manifolds which share a common boundary $\partial_2 X := \partial(\partial_1^+ X) = \partial(\partial_1^- X)$. Evidently, $\partial_2 X$ is the locus where v is *tangent* to the boundary $\partial_1 X$.

Morse has noticed that, for a generic vector field v , the tangent locus $\partial_2 X$ inherits a similar structure in connection to $\partial_1^+ X$, as $\partial_1 X$ has in connection to X (see [Mo]). That is, v gives rise to a partition $\partial_2^+ X \cup \partial_2^- X$ of $\partial_2 X$ into two sets: the locus $\partial_2^+ X$, where the field is directed inward of $\partial_1^+ X$, and $\partial_2^- X$, where it is directed outward of $\partial_1^+ X$. Again, let us assume that v , viewed as a section of the quotient line bundle $T(\partial_1 X)/T(\partial_2 X)$ over $\partial_2 X$, is transversal to its zero section.

For generic fields, this structure replicates itself: the cuspidal locus $\partial_3 X$ is defined as the locus where v is tangent to $\partial_2 X$; $\partial_3 X$ is divided into two manifolds, $\partial_3^+ X$ and $\partial_3^- X$. In $\partial_3^+ X$, the field is directed inward of $\partial_2^+ X$, in $\partial_3^- X$, outward of $\partial_2^+ X$. We can repeat this construction until we reach the zero-dimensional stratum $\partial_{n+1} X = \partial_{n+1}^+ X \cup \partial_{n+1}^- X$.

These considerations motivate

Definition 1.1. *We say that a smooth field v on X is boundary generic if:*

- $v|_{\partial X} \neq 0$,
- v , viewed as a section of the tangent bundle $T(X)$, is transversal to its zero section,
- for each $j = 1, \dots, n+1$, the v -generated stratum $\partial_j X$ is a smooth submanifold of $\partial_{j-1} X$,
- the field v , viewed as section of the quotient 1-bundle

$$T_j^\nu := T(\partial_{j-1} X)/T(\partial_j X) \rightarrow \partial_j X,$$

is transversal to the zero section of T_j^ν for all $j > 0$.

We denote by $\mathcal{V}^\dagger(X)$ the space of all boundary generic fields on X . □

Thus a boundary generic vector field v on X gives rise to two Morse stratifications:

$$(1.1) \quad \begin{aligned} \partial X &:= \partial_1 X \supset \partial_2 X \supset \dots \supset \partial_{n+1} X, \\ X &:= \partial_0^+ X \supset \partial_1^+ X \supset \partial_2^+ X \supset \dots \supset \partial_{n+1}^+ X \end{aligned}$$

, the first one by closed submanifolds, the second one—by compact ones. Here $\dim(\partial_j X) = \dim(\partial_j^+ X) = n+1-j$. For simplicity, the notations “ $\partial_j^\pm X$ ” do not reflect the dependence of these strata on the vector field v . When the field varies, we use a more accurate notation “ $\partial_j^\pm X(v)$ ”.

When v is nonsingular on $\partial_1 X$, we can extend it into a larger manifold \hat{X} so that \hat{X} properly contains X and the extension \hat{v} remains nonsingular in the vicinity of $\partial_1 X \subset \hat{X}$. Throughout this text, we treat the pair (\hat{X}, \hat{v}) as a germ which extends (X, v) .

At each point $x \in \partial_1 X$, the $(-\hat{v})$ -flow defines the germ of the projection $p_x : \hat{X} \rightarrow S_x$, where S_x is a local section of the \hat{v} -flow which is transversal to it. The projection is considered at each point of $\partial_1 X \subset \hat{X}$. When \hat{v} is a gradient-like field for a function $\hat{f} : \hat{X} \rightarrow \mathbb{R}$, we can choose the germ of the hypersurface $f^{-1}(f(x))$ for the role of S_x .

For boundary generic vector field v , we associate an ordered sequence of multiplicities with each trajectory γ , such that $\gamma \cap \partial_1 X$ is a finite set. In fact, for any $v \in \mathcal{V}^\dagger(X)$, the intersection $\{\alpha_i\} := \gamma \cap \partial_1 X$ is automatically a finite set. For traversing (see Definition 4.6 from [K1]) generic fields, the points $\{a_i\}$ of the intersection $\gamma \cap \partial_1 X$ are ordered by the field-oriented trajectory γ , and the index i reflects this ordering.

Definition 1.2. *Let $v \in \mathcal{V}^\dagger(X)$ be a generic field. Let γ be a v -trajectory which intersects the boundary $\partial_1 X$ at a finite number of points $\{a_i\}$. Each point a_i belongs to a unique pure stratum $\partial_{j_i} X^\circ$.*

The multiplicity $m(\gamma)$ of γ is defined by the formula

$$(1.2) \quad m(\gamma) = \sum_i j_i$$

The reduced multiplicity $m'(\gamma)$ of γ is defined by the formula

$$(1.3) \quad m'(\gamma) = \sum_i (j_i - 1)$$

, and the virtual multiplicity $\mu(\gamma)$ of γ is defined by

$$(1.4) \quad \mu(\gamma) = \sum_i \left\lceil \frac{j_i}{2} \right\rceil$$

, where $\lceil \cdot \rceil$ denotes the integral part function. □

For an open and dense subspace $\mathcal{V}^\dagger(X)$ of $\mathcal{V}^\dagger(X)$, one can interpret $\mu(\gamma)$ as the maximal number of tangency points that any trajectory γ' in the vicinity of γ may have (see Theorem 3.4 from [K2]).

When $v \in \mathcal{V}^\dagger(X)$, each set $\partial_j X(v)$ is a manifold.

Let $\partial_j X(v)^\circ := \partial_j X(v) \setminus \partial_{j+1} X(v)$ denotes the pure Morse stratum.

The following lemma (see Lemma 3.1 from [K2] and [Morin]) provides us with an analytic description of the Morse strata.

Lemma 1.1. *Assume that v is a boundary generic field. Denote by γ_a the \hat{v} -trajectory through a point $a \in X$. If $a \in \partial_k X(v)^\circ$ then in the vicinity of a in \hat{X} , there exists a coordinate system (u, x) , where $u \in \mathbb{R}$ and $x \in \mathbb{R}^n$, so that:*

- *each v -trajectory γ is defined by an equation $\{x = \text{const}\}$,*
- *the boundary $\partial_1 X$ is defined by the equation*

$$(1.5) \quad u^k + \sum_{j=0}^{k-2} x_j u^j = 0$$

- *each v -trajectory γ hits only some strata $\{\partial_j X(v)^\circ\}_{j \in J(a)}$ in such a way that*

$$\sum_{j \in J(a)} j \leq k, \quad \text{and} \quad \sum_{j \in J(a)} j \equiv k \pmod{2}.$$

□

The introduction of the next class $\mathcal{V}^\ddagger(X)$ of vector fields is inspired by the singularity theory of so called *Boardman maps* with normal crossings (see [Bo], [GG]).

Consider the collection of tangent spaces $\{T_{a_i}(\partial_{j_i} X^\circ)\}_i$ to the pure strata $\{\partial_{j_i} X^\circ\}_i$ that have a non-empty intersection with a given trajectory γ . By Lemma 1.1, each space $T_{a_i}(\partial_{j_i} X^\circ)$ is transversal to the curve γ .

Let S be a local section of the \hat{v} -flow at some point $a_\star \in \gamma$ and let T_\star be the space tangent to S at a_\star . Each space $T_{a_i}(\partial_{j_i} X^\circ)$, with the help of the \hat{v} -flow, determines a vector

subspace $T_i = T_i(\gamma)$ in T_\star . It is the image of the tangent space $T_{a_i}(\partial_j X^\circ)$ under the composition of two maps: (1) the differential of the flow-generated diffeomorphism that maps a_i to a_\star and (2) the linear projection $T_{a_\star}(X) \rightarrow T_\star$ whose kernel is generated by $v(a_\star)$.

For a traversing v and a majority of trajectories, we can choose the space $T_{a_\star}(\partial_1^+ X)$ for the role of T_\star , where a_\star is the lowest point of $\gamma \cap \partial_1 X$.

The configuration $\{T_i\}$ of *affine* subspaces $T_i \subset T_\star$ is called *generic* (or *stable*) when all the multiple intersections of spaces from the configuration have the least possible dimensions, consistent with the dimensions of $\{T_i\}$. In other words,

$$\text{codim}(\bigcap_s T_{i_s}, T_\star) = \sum_s \text{codim}(T_{i_s}, T_\star)$$

for any subcollection $\{T_{i_s}\}$ of spaces from the list $\{T_i\}$.

Consider the case when $\{T_i\}$ are *vector* subspaces of T_\star . If we interpret each T_i as the kernel of a linear epimorphism $\Phi_i : T_\star \rightarrow \mathbb{R}^{n_i}$, then the property of $\{T_i\}$ being generic can be reformulated as the property of the direct product map $\prod_i \Phi_i : T_\star \rightarrow \prod_i \mathbb{R}^{n_i}$ being an epimorphism. In particular, for a generic configuration of affine subspaces, if a point belongs to several T_i 's, then the sum of their codimensions n_i does not exceed the dimension of the ambient space T_\star .

The definition below resembles the ‘‘Normal Crossing Condition’’ imposed on Boardman maps between smooth manifolds (see [GG], page 157, for the relevant definitions). In fact, for traversing generic fields v , the v -flow delivers germs of Boardman maps $p(v, \gamma) : \partial_1 X \rightarrow \mathbb{R}^n$, available in the vicinity of every trajectory γ .

Definition 1.3. *We say that a traversing field v on X is transversally generic if:*

- *the field is boundary generic in the sense of Definition 1.1,*
- *for each v -trajectory $\gamma \subset X$ (not a singleton), the collection of subspaces $\{T_i(\gamma)\}_i$ is generic in T_\star : that is, the obvious quotient map $T_\star \rightarrow \prod_i (T_\star / T_i(\gamma))$ is surjective.*

We denote by $\mathcal{V}^\dagger(X)$ the space of all transversally generic fields on X . □

Remark 1.1. In particular, the second bullet of the definition implies the inequality

$$\sum_i \text{codim}(T_i(\gamma), T_\star) \leq \dim(T_\star) = n.$$

In other words, for transversally generic fields, the reduced multiplicity of each trajectory γ satisfies the inequality

$$(1.6) \quad m'(\gamma) = \sum_i (j_i - 1) \leq n.$$

□

The following key lemma (see Lemma 3.4 from [K2]) provides us a semi-local analytic description of the transversally generic fields.

Lemma 1.2. *Let v be a traversing generic field on X and γ its trajectory such that the intersection $\gamma \cap \partial_1 X$ is a union of several points $a_i \in \partial_{j_i} X(v)^\circ$.*

Then γ has a \hat{v} -adjusted neighborhood V with a special system of coordinates

$$(u, \underbrace{x_{10}, \dots, x_{1j_1-2}}, \dots, \underbrace{x_{i0}, \dots, x_{ij_i-2}}, \dots, \underbrace{x_{p0}, \dots, x_{pj_p-2}}, \underbrace{y_1, \dots, y_{n-m'(\gamma)}})$$

such that:

- $\{u = \text{const}\}$ defines a transversal section of the \hat{v} -flow,
- each \hat{v} -trajectory in V is produced by fixing all the coordinates $\{x_{il}\}$ and $\{y_k\}$,
- there is $\epsilon > 0$ such that $V \cap \partial_1 X \subset \coprod_i V_i$, where $V_i := u^{-1}((\alpha_i - \epsilon, \alpha_i + \epsilon)) \cap V$, and $\alpha_i = u(a_i)$,
- the intersection $V \cap \partial_1 X$ is given by the equation

$$(1.7) \quad \prod_i [(u - \alpha_i)^{j_i} + \sum_{l=0}^{j_i-2} x_{i,l}(u - \alpha_i)^l] = 0.$$

□

2. BIFURCATIONS OF THE REAL POLYNOMIAL DIVISORS — THE COMBINATORICS OF TANGENCY FOR TRAVERSALLY GENERIC FLOWS

In [K2], we have seen evidence that, for smooth transversally generic fields v on a compact manifold X , the combinatorial patterns of tangency along v -trajectories resemble the combinatorics of divisors in \mathbb{R} , produced by real polynomials of an even degree $d \leq 2 \dim(X)$ (see Lemma 3.4, Theorems 3.1, and Theorem 3.5 from [K2]). Now we are going to devote time to somewhat involved investigations of this combinatorics.

For any polynomial $P(z)$ with real coefficients, we denote by $D_{\mathbb{R}}(P)$ its divisor in \mathbb{R} , and by $D_{\mathbb{C}}(P)$ its complex conjugation-invariant divisor in \mathbb{C} . Points in $\text{sup}(D_{\mathbb{R}}(P))$, the support of $D_{\mathbb{R}}(P)$, inherit the natural order from \mathbb{R} . The set $\text{sup}(D_{\mathbb{C}}(P)) \subset \mathbb{C}$ is invariant under the complex conjugation.

Let \mathcal{D}_d denote the space of all divisors D in \mathbb{R} of degree $|D| = d$. The space \mathcal{D}_d can be identified with the d -th symmetric power $\text{Sym}^d(\mathbb{R})$ of the real number line \mathbb{R} . Alternatively, \mathcal{D}_d can be introduced as the domain Π_d in \mathbb{R}^d given by the inequalities $x_1 \leq x_2 \leq \dots \leq x_d$ imposed on the coordinates (x_1, x_2, \dots, x_d) .

The divisors $D \in \mathcal{D}_d$ have *combinatorial models* represented by maps

$$\omega_D : \{1, 2, 3, \dots\} \rightarrow \{0, 1, 2, 3, \dots\}$$

, where $\omega_D(i) \geq 1$ is the multiplicity of the i -th point in $\text{sup}(D)$, so that $\sum_i \omega_D(i) = d$.

Let \mathbb{N} be the set of all natural numbers and \mathbb{Z}_+ the set of all non-negative integers. Consider the set Ω of all maps $\omega : \mathbb{N} \rightarrow \mathbb{Z}_+$ with finite support and such that $\omega(i) \neq 0$ implies $\omega(j) \neq 0$ for all $j < i$.

Consider a real polynomial P of degree d and its complex divisor $D_{\mathbb{C}}(P)$. There is $\epsilon > 0$ such that the ϵ -neighborhood U_ϵ of the support $\text{sup}(D_{\mathbb{C}}(P)) \subset \mathbb{C}$ is a union of *disjoint*

open disks. Then any real d -polynomial Q , sufficiently close to P , will have all its roots residing in U_ϵ .

The radial contraction of each disk from U_ϵ to its center commutes with the complex conjugation in \mathbb{C} and defines a conjugation-equivariant deformation of the divisor $D_{\mathbb{C}}(Q)$ into the divisor $D_{\mathbb{C}}(P)$. This produces a deformation of Q to P , a curve Q_s , $s \in [0, 1]$, in the space of real polynomials of degree d . At all stages of this deformation, but the last one ($s = 1$), the divisor $D_{\mathbb{C}}(Q_s)$ has the same multiplicity pattern as the one of $D_{\mathbb{C}}(Q)$; moreover, the divisor $D_{\mathbb{R}}(Q_s)$ has the same \mathbb{R} -ordered multiplicity pattern as the one of $D_{\mathbb{R}}(Q)$.

Now imagine this deformation process from the viewpoint of an observer residing in \mathbb{R} , so that morphings of all the roots residing in $\mathbb{C} \setminus \mathbb{R}$ are “invisible”.

Deformations of P within the space of real polynomials of degree $\deg(P)$ change its real divisor $D_{\mathbb{R}}(P)$ by sequences of two *elementary operations* and their inverses:

- (1) merging of two adjacent points from the support $\text{sup}(D_{\mathbb{R}}(P))$ (their multiplicities add up);
- (2) inserting a point of multiplicity 2 to the set $\mathbb{R} \setminus \text{sup}(D_{\mathbb{R}}(P))$.

The second elementary operation corresponds to a pair of simple complex-conjugate roots merging at a point of $\mathbb{R} \subset \mathbb{C}$.

These operations have combinatorial analogues. We define the elementary merge operation $M_j : \Omega \rightarrow \Omega$ by the formula

$$(2.1) \quad \begin{aligned} M_j(\omega)(i) &= \omega(i) \text{ for all } i < j, \\ M_j(\omega)(j) &= \omega(j) + \omega(j+1), \\ M_j(\omega)(i) &= \omega(i+1) \text{ for all } i > j+1, \end{aligned}$$

where $\omega \in \Omega$ and $1 \leq j \leq |\text{sup}(\omega)|$.

Define the elementary insert operation $I_j : \Omega \rightarrow \Omega$ by the formula

$$(2.2) \quad \begin{aligned} I_j(\omega)(i) &= \omega(i) \text{ for all } i < j, \\ I_j(\omega)(j) &= 2, \\ I_j(\omega)(i) &= \omega(i-1) \text{ for all } i > j \geq 1, \end{aligned}$$

and

$$(2.3) \quad \begin{aligned} I_0(\omega)(1) &= 2 \\ I_0(\omega)(i) &= \omega(i-1), \text{ for all } i > 1. \end{aligned}$$

We also allow for the merge of several adjacent points in $\text{sup}(D_{\mathbb{R}}(P))$ and for the merge of groups of conjugate roots (of any multiplicity) at a point(s) of \mathbb{R} . However, we tend to view these morphings as compositions of the elementary operations $\{M_j\}$ and $\{I_j\}$.

Note that the *parity* of the degree $\deg(D_{\mathbb{R}}(P))$ is preserved under the merge and insert operations.

Guided by Definition 1.2, we introduce combinatorial analogues of the quantities from that definition:

- the l_1 -norm $|\omega|$ of ω by the formula $\sum_i \omega(i)$,¹
- the *reduced norm* $|\omega|'$ of ω , by the formula $\sum_i (\omega(i) - 1)$,
- the *virtual multiplicity* $\mu(\omega)$ of ω , by the formula $\sum_i \lceil \omega(i)/2 \rceil$, where $\lceil \cdot \rceil$ denotes the integral part of a real number.

Now we are in position to define, via the merge and insert operations, a *partial order* " \succ " in the set Ω :

Definition 2.1. For $\omega_1, \omega_2 \in \Omega$, we write $\omega_1 \succ \omega_2$ if ω_2 can be obtained from ω_1 by a sequence of merge operations $\{M_j\}$ and insert operations $\{I_j\}$ as in (2.1)-(2.3). \square

Let $\Omega_d \subset \Omega$ denote the set of all ω 's such that $|\omega| = d$. Let $\Omega_{\leq d} \subset \Omega$ denote the set of all ω 's such that $|\omega| \leq d$ and $|\omega| \equiv d \pmod{2}$. The set $\Omega_{\leq d}$ inherits its partial order from the ambient poset (Ω, \succ) .

For all our applications we will need only the case of an even d .

We can give an interpretation to the poset (Ω, \succ) in the spirit of category theory. In this interpretation, the elements of Ω become objects of some category $\mathbf{\Omega}$ so that the relation $\omega_1 \succ \omega_2$ is transformed into the property $Mor(\omega_1, \omega_2) \neq \emptyset$.

The following definition mimics the map of conjugation-invariant divisors $D_{\mathbb{C}}(Q) \rightarrow D_{\mathbb{C}}(P)$, induced by the radial retraction of the ϵ -neighborhood $U_\epsilon(\text{sup}(D_{\mathbb{C}}(P)))$ to its core, $\text{sup}(D_{\mathbb{C}}(P))$, as observed from within $\mathbb{R} \subset \mathbb{C}$.

Definition 2.2. For any two elements $\omega_1, \omega_2 \in \Omega$ we define the set $Mor(\omega_1, \omega_2)$ as the set of maps $\alpha : \text{sup}(\omega_1) \rightarrow \text{sup}(\omega_2)$ such that:

- (1) for each pair $i < i'$ in $\text{sup}(\omega_1)$, $\alpha(i) \leq \alpha(i')$,
- (2) $\sum_{j \in \alpha^{-1}(i)} \omega_1(j) \leq \omega_2(i)$ for all $i \in \text{sup}(\omega_2)$,²
- (3) $\sum_{j \in \alpha^{-1}(i)} \omega_1(j) \equiv \omega_2(i) \pmod{2}$ for all $i \in \text{sup}(\omega_2)$.³ \square

For example, $Mor((121), (11)) = \emptyset$, while $Mor((11), (121))$ consists a single injective map.

Examining properties (1)-(3) above, we see that there is a natural pairing

$$Mor(\omega_1, \omega_2) \times Mor(\omega_2, \omega_3) \rightarrow Mor(\omega_1, \omega_3)$$

defined by the composition of maps, provided $Mor(\omega_1, \omega_2) \neq \emptyset$ and $Mor(\omega_2, \omega_3) \neq \emptyset$.

Lemma 2.1. In the category $\mathbf{\Omega}$, the set $Mor(\omega_1, \omega_2) \neq \emptyset$, if and only if, $\omega_1 \succeq \omega_2$ in the poset (Ω, \succ) .

Proof. Any elementary operation M_j in (2.1) gives rise to an element $\mu_j \in Mor(\omega, M_j(\omega))$ which maps the pair $j, j+1 \in \text{sup}(\omega)$ to the single element $j \in \text{sup}(M_j(\omega))$; the rest of the elements in $\text{sup}(\omega)$ are mapped bijectively by μ_j . Evidently, the properties (1)-(3) in Definition 2.2 are satisfied. Similarly, any elementary operation I_j in (2.2) gives rise to an

¹This number represents $\deg(D_{\mathbb{R}}(P))$.

²This restriction is vacuous when $\alpha^{-1}(i) = \emptyset$.

³In particular, if $\alpha^{-1}(i) = \emptyset$, then $\omega_2(i) \equiv 0 \pmod{2}$.

element $\nu_j \in \text{Mor}(\omega, l_j(\omega))$ which maps $\text{sup}(\omega)$ bijectively and in a monotone fashion to $\text{sup}(l_j(\omega))$ so that $j \in \text{sup}(l_j(\omega))$ is the only element that is not in the image of ν_j . Again, the properties (1)-(3) in Definition 2.2 are satisfied.

Therefore if two elements, ω_1 and ω_2 , are linked by a sequence of elementary operations of the types M_j and l_k , then there exists an element $\alpha \in \text{Mor}(\omega_1, \omega_2)$ which is obtained by composing the chain of corresponding maps μ_j and ν_k . Hence, $\text{Mor}(\omega_1, \omega_2) \neq \emptyset$.

On the other hand, if $\text{Mor}(\omega_1, \omega_2) \neq \emptyset$, then any $\alpha \in \text{Mor}(\omega_1, \omega_2)$ can be obtained in such a way via elementary operations. Indeed, for each $i \in \text{sup}(\omega_2)$, consider the set $\alpha^{-1}(i)$. Put

$$a_i = \omega_2(i) - \sum_{j \in \alpha^{-1}(i)} \omega_1(j).$$

By Definition 2.2, $a_i \geq 0$ and $\alpha_i \equiv 0 \pmod{2}$. We apply a sequence of $a_i/2$ insert operations l_k to ω_1 , localized to the set $\alpha^{-1}(i)$. They will add $a_i/2$ copies of 2's to the sequence ω_1 . The location of these 2's relative to the elements of $\alpha^{-1}(i)$ is unimportant. The resulting ω'_1 now has the property $\text{Mor}(\omega_1, \omega'_1) \neq \emptyset$. These insertions $\{l_k\}$ define a map $\gamma \in \text{Mor}(\omega_1, \omega'_1)$. Next we merge all the original elements of the set $\alpha^{-1}(i)$ and the locations of newly inserted 2's together into a singleton i_* by a sequence of elementary merges $\{M_j\}$. Again, the order in which the elementary merges are performed is unimportant. The resulting ω''_1 is such that there exists $\delta \in \text{Mor}(\omega_1, \omega''_1)$ with the property

$$\sum_{j \in \delta^{-1}(i_*)} \omega_1(j) = \omega''_1(i_*) = \omega_2(i).$$

Finally, we apply this procedure to every element $i \in \text{sup}(\omega_2)$. The result of these operations transforms ω_1 into ω_2 by a sequence of elementary operations. Therefore, $\omega_1 \succ \omega_2$ in the poset Ω . \square

The polynomial inequality $P(z) \leq 0$ splits the support $\text{sup}(D_{\mathbb{R}}(P))$ into a number of disjoint sets: each set is formed by the maximal string of consecutive roots so that, in the closed interval bounded by the maximal and the minimal root from the string, the inequality $P(z) \leq 0$ is valid. For a polynomial of an even degree, each maximal string of roots either is a singleton whose multiplicity is even, or a sequence whose maximal and minimal elements have odd multiplicities, while the rest of roots have even multiplicities. These observations motivate the following combinatorial models.

Definition 2.3. Let Ω^\bullet denote the set of maps $\omega \in \Omega$ that satisfy the following properties:

- either
 - (1) $\omega(1), \omega(q)$ are odd numbers, where $q = |\text{sup}(\omega)|$, and
 - (2) $\omega(i)$ is even for $1 < i < q$;
- or $\text{sup}(\omega) = \{1\}$, and $\omega(1) \equiv 0 \pmod{2}$.

\square

A priori q , the cardinality of the support of $\omega \in \Omega^\bullet$, is not fixed.

Definition 2.4. Let $\Omega_{\bullet, d}^\bullet$ denote the set of maps $\omega \in \Omega^\bullet$ such that $|\omega|' := \sum_i (\omega(i) - 1) = d$.

Let

$$\Omega_{[k,d]}^\bullet := \coprod_{k \leq j \leq d} \Omega_j^\bullet.$$

We also will use the shorter notation “ $\Omega_{[d]}^\bullet$ ” for the set $\Omega_{[0,d]}^\bullet$. \square

Recall that on a $(n+1)$ -manifold X , any trajectory γ of a boundary generic field $v \in \mathcal{V}^\dagger(X)$ gives rise to an element $\omega_\gamma \in \Omega^\bullet$ according to the rule: $\omega_\gamma(i) = m(a_i)$, the multiplicity of the i -th point a_i in the v -ordered set $\gamma \cap \partial_1 X$ ⁴. By Theorem 3.5 from [K2], for any transversally generic field $v \in \mathcal{V}^\dagger(X)$, we get $|\omega_\gamma|' \leq n = \dim(\partial_1 X)$, so that $\omega_\gamma \in \Omega_{[n]}^\bullet$.

In the end of the proof of Theorem 3.5 from [K2], we have established the following proposition:

Lemma 2.2. *For $\omega \in \Omega_{[n]}^\bullet$, we have $|\omega| := \sum_i \omega_\gamma(i) \leq 2n + 2$.* \square

We are going to introduce a partial order among the elements of the set $\Omega_{[d]}^\bullet$, which will match the changing geometry of orbits for transversally generic fields. Crudely, the order is induced from the ambient poset $\Omega \supset \Omega_{[d]}^\bullet$, but then enhanced. For a more accurate description, we turn to few auxiliary combinatorial constructions.

Given any $\omega \in \Omega$, we would like to “chop it” into a number of “strings” and “atoms”: each string belongs to some Ω_d^\bullet as in the first bullet of the Definition 2.3, while each atom has a singleton for its support and takes there an even value (see Fig. 1). Prior to defining this canonical deconstruction $\Xi(\omega)$ of ω , we need to introduce some notations.

Let $\omega \in \Omega$ be such that $|\omega| \equiv 0 \pmod{2}$. Denote by $\sup_{\text{odd}}(\omega)$ the points $l \in \mathbb{N}$ in the support of ω such that $\omega(l) \equiv 1 \pmod{2}$ and by $\sup_{\text{ev}}(\omega)$ the points l in the support of ω such that $\omega(l) \equiv 0 \pmod{2}$. In the geometrical context of traversing flows, the number $|\sup_{\text{odd}}(\omega)|$ is even. We count the elements of $\sup_{\text{odd}}(\omega)$ as they appear in the list; some of them acquire odd numerals, others acquire even ones. A *string* in $A \subset \sup(\omega)$ is formed by all the elements that are bounded on the left by an element of $i \in \sup_{\text{odd}}(\omega)$ with an odd numeral and on the right by the next element $j \in \sup_{\text{odd}}(\omega)$ (it has an even numeral attached to it). In other words, a string $A := [i, j]$ is formed by such i, j and all the elements from $\sup_{\text{ev}}(\omega)$ that lie in-between. The points from $\sup_{\text{ev}}(\omega)$ that do not belong to any string are called “atoms”. We can compute the values of ω at the elements of each string A as well as at each atomic support. This gives rise to a unique ordered sequence $\Xi(\omega)$ of strings, interrupted by a number of atoms.

Definition 2.5. *We define the partial order “ \prec_\bullet ” in Ω^\bullet as follows: for $\omega_1, \omega_2 \in \Omega^\bullet$, we write “ $\omega_1 \prec_\bullet \omega_2$ ” if ω_2 occurs as a string or an atom from the list $\Xi(\omega)$ for some $\omega \succ \omega_1$, where $\omega \in \Omega$ and the last ordering is considered in the poset (Ω, \succ) .* \square

Example 2.1. Consider the poset $(\Omega_{[0,3]}^\bullet, \prec_\bullet)$ shown in Fig. 2. Then $(1, 4, 1) \prec_\bullet (3, 1)$ since $(3, 1)$ is present as a string in $(1, 1, 3, 1) \succ (1, 4, 1)$, the order “ \succ ” being the one from the poset $\Omega_{[0,8]}$. \square

⁴that is, $a_i \in \partial_{m(a_i)} X(v)^\circ$.

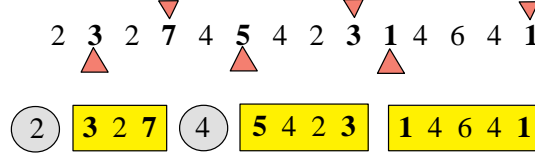


FIGURE 1. An example of the deconstruction $\omega \Rightarrow \Xi(\omega)$. The strings are boxed, the atoms are circled.

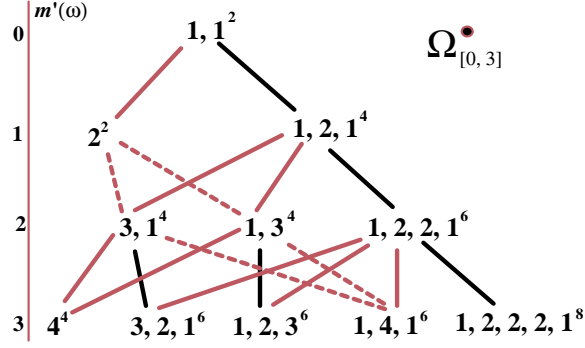


FIGURE 2. The poset $(\Omega_{[0,3]}^\bullet, \prec_\bullet)$ with the "height" function $m'(\omega) := |\omega|'$. The bold dark lines indicate the insert operations, the lighter bold lines the merge operations. The dotted lines represent the order relations, as in Definition 2.5, that are not directly induced from the poset $\Omega_{[0,8]} \supset \Omega_{[0,3]}^\bullet$. The upper index next to ω shows the value of the norm $|\omega|$.

Lemma 2.3. *For any d , the sub-poset of $\Omega_{[d]}^\bullet$, defined by the constraint $|\omega|' < d$, is canonically isomorphic to the poset $\Omega_{[d-1]}^\bullet$. As a result, the direct limit*

$$\lim_{d \rightarrow +\infty} \Omega_{[d]}^\bullet = \Omega^\bullet$$

as posets.

Proof. Let $\omega_1 \succ_\bullet \omega_2$ in $\Omega_{[d]}^\bullet$. This implies that ω_1 is string or an atom in some $\omega \succ \omega_2$ in (Ω, \succ) . If $|\omega_2|' < d$, then $|\omega|' < d$ as well. In turn, $|\omega|' < d$ implies that $|\omega_1|' < d$. Therefore $\omega_1 \succ_\bullet \omega_2$ in $\Omega_{[d-1]}^\bullet$ as well. \square

Similar to the poset (Ω, \succ) , the poset $(\Omega^\bullet, \succ_\bullet)$ allows for an interpretation in terms of a category theory $\mathbf{\Omega}^\bullet$ whose objects are elements of Ω^\bullet .

Definition 2.6. *For any two elements $\omega_1, \omega_2 \in \Omega^\bullet$ we define $\text{Mor}^\bullet(\omega_1, \omega_2)$ as the set of maps $\alpha : \text{sup}(\omega_1) \rightarrow \text{sup}(\omega_2)$ such that:*

- (1) *for each pair $i < i'$ in $\text{sup}(\omega_1)$, $\alpha(i) \leq \alpha(i')$,*

(2) $\sum_{j \in \alpha^{-1}(i)} \omega_1(j) \leq \omega_2(i)$ for all $i \in \text{sup}(\omega_2)$ ⁵. \square

Lemma 2.4. *In the category Ω^\bullet , the set $\text{Mor}_\bullet(\omega_1, \omega_2) \neq \emptyset$, if and only if, $\omega_1 \succeq_\bullet \omega_2$ in the poset $(\Omega_\bullet, \succ_\bullet)$.*

Proof. If $\omega_1 \succeq_\bullet \omega_2$, then by the definition of the order \succ_\bullet , there exists an element $\tilde{\omega}_1 \in \Omega$ such that $\tilde{\omega}_1 \succ \omega_2$, and ω_1 is a string or an atom in $\tilde{\omega}_1$. By Lemma 2.1, $\text{Mor}(\tilde{\omega}_1, \omega_2) \neq \emptyset$. When we restrict a map $\tilde{\alpha} \in \text{Mor}(\tilde{\omega}_1, \omega_2)$ to the subset $\text{sup}(\omega_1) \subset \text{sup}(\tilde{\omega}_1)$ we produce a map $\alpha : \text{sup}(\omega_1) \rightarrow \text{sup}(\omega_2)$. Under this restriction, the properties (1) and (2) from Definition 2.2, valid for $\tilde{\alpha}$, are evidently preserved for its restriction α : the support of a string consists of consecutive indices, and $\tilde{\alpha}^{-1}(i) \supset \alpha^{-1}(i)$ for all $i \in \text{sup}(\omega_2)$, while the property (3) could be violated. Thus $\alpha \in \text{Mor}^\bullet(\omega_1, \omega_2)$, a nonempty set.

On the other hand, if $\text{Mor}^\bullet(\omega_1, \omega_2) \neq \emptyset$, then it is possible to construct $\tilde{\omega}_1 \in \Omega$ such that $\text{Mor}(\tilde{\omega}_1, \omega_2) \neq \emptyset$. Indeed, consider some $\alpha : \text{sup}(\omega_1) \rightarrow \text{sup}(\omega_2)$ having properties (1) and (2) from Definition 2.6. If, in addition, property (3) from Definition 2.2 holds, we are done. Note that this parity property can only be violated if one or both ends $\{1\}, \{q\}$ in the support of the string ω_1 are mapped by α to the location $\alpha(1)$ or $\alpha(q)$ such that $\omega_2(\alpha(1)) \equiv 0 \pmod{2}$, or $\omega_2(\alpha(q)) \equiv 0 \pmod{2}$.

Let us consider the case $\omega_2(\alpha(1)) \equiv 0 \pmod{2}$. The monotonicity of α implies that the minimum element $\{1_\star\}$ in the support of ω_2 is not in the image of α . We can append a string $\omega' = (1, 1)$ below (to the left) of the string ω_1 to form a new element $\tilde{\omega}_1 \in \Omega$ (not a string!) so that $\text{sup}(\tilde{\omega}_1) = \text{sup}((1, 1)) \sqcup \text{sup}(\omega_1)$. Then we extend the map α to a map $\tilde{\alpha} : \text{sup}((1, 1)) \sqcup \text{sup}(\omega_1) \rightarrow \text{sup}(\omega_2)$ by sending the first element of the extended support to the minimal element $\{1_\star\} \in \text{sup}(\omega_2)$, and the second and the third elements both to $\alpha(1)$ (note that the third element comes from the element $\{1\} \in \text{sup}(\omega_1)$). This will repair the parity defect of the original α . A similar treatment applies when $\omega_2(\alpha(q)) \equiv 0 \pmod{2}$. Thus, $\text{Mor}^\bullet(\omega_1, \omega_2) \neq \emptyset$ implies that $\text{Mor}(\tilde{\omega}_1, \omega_2) \neq \emptyset$.

By Lemma 2.1, we get that $\tilde{\omega}_1 \succ \omega_2$ in (Ω, \succ) . Therefore, $\text{Mor}^\bullet(\omega_1, \omega_2) \neq \emptyset$ implies that $\omega_1 \succ_\bullet \omega_2$. \square

The following lemma can be viewed as an a posteriori justification for introducing the poset $(\Omega^\bullet, \succ_\bullet)$. Here, for a boundary generic vector field $v \in \mathcal{V}^\dagger(X)$ and its trajectory γ , we view the set $\gamma \cap \partial_1 X$, together with the multiplicities attached to its points, as a divisor D_γ on γ .

Lemma 2.5. *Let $v \in \mathcal{V}^\dagger(X)$. If a sequence $\{x_k \in X\}_k$ converges to a point x , then*

$$|D_{\gamma(x)}|' \geq \overline{\lim}_{k \rightarrow +\infty} |D_{\gamma(x_k)}|'$$

, i.e. the reduced norm $|D_{\gamma(\sim)}|'$ is a upper semi-continuous function on X . Moreover, if the divisors $\{D_{\gamma(x_k)}\}_k$ all share a combinatorial type ω , and $\omega_1 \neq \omega$ is the combinatorial type of $D_{\gamma(x)}$, then we get: $\omega \succ_\bullet \omega_1$, $|\omega_1| \geq |\omega|$, and $|\omega_1|' > |\omega|'$.

⁵This restriction is vacuous when $\alpha^{-1}(i) = \emptyset$.

Proof. The trust of the argument is that any merge or insert operations (see (2.1)-(2.3)) that change the combinatorics of D_γ cannot decrease the multiplicity $m(\gamma)$, as well as the reduced multiplicity $m'(\gamma)$ (defined in (1.2)-(1.4)). In fact, any such operation does increase $m'(g)$.

In the argument below, we use our understanding of local models (1.5) (see Lemma 1.1) of fields $v \in \mathcal{V}^\dagger(X)$ in the vicinity of $\partial_1 X$. They imply that, as we vary γ , its divisor D_γ (with the support $\gamma \cap \partial_1 X$) can only change locally as the real divisors of the family of u -polynomials in (1.5) do: that is, via resolutions that either preserve their degree, or drop it by an even number. By the same token, as $\{x_k\}$ converge to x , in the vicinity of x , the divisor $D_{\gamma(x)}$ is obtained from $\{D_{\gamma(x_k)}\}$ only via: (1) merging adjacent distinct points from $\text{sup}(D_{\gamma(x_k)})$, or (2) by inserting new points of even multiplicity, or 3) by increasing the odd multiplicity of the end point from $\text{sup}(D_{\gamma(x_k)})$ by an odd number (this happens when the ends of two distinct trajectories merge at x).

Let us take a closer look at these mechanisms.

By Lemma 1.1, for a boundary generic $v \in \mathcal{V}^\dagger(X)$, the multiplicity of each point from $\gamma \cap \partial_1 X$ does not exceed $n+1$ for all γ 's. Moreover, by the same lemma and a compactness argument, no trajectory γ has an infinite set $\gamma \cap \partial_1 X$. Thus the multiplicity $m(\gamma) < \infty$ for each γ .

Consider any infinite set of trajectories $\{\gamma_k\}_k$ such that the set of multiplicities $\{m(\gamma_k)\}_k$ is unbounded. Since the multiplicity of each point of tangency is uniformly bounded, this implies that the set

$$A := \cup_{1 \leq k < \infty} (\gamma_k \cap \partial_1 X)$$

is infinite and $\lim_{k \rightarrow \infty} |\gamma_k \cap \partial_1 X| = \infty$. Let y be a limit point for A . Consider the trajectory γ_y through y . Using the compactness of γ_y and the local models from Lemma 1.1, γ_y has a neighborhood in which any trajectory $\tilde{\gamma}$ has a uniformly bounded cardinality of the set $\tilde{\gamma} \cap \partial_1 X$. This contradicts to the assumption that $\lim_{k \rightarrow \infty} |\gamma_k \cap \partial_1 X| = \infty$. Therefore there is a number d so that, for every trajectory γ in X , its combinatorial type belongs to the set $\Omega_{[d]}$.

Consider the combinatorial types of $\{D_{\gamma(x_k)}\}_k$, a finite set. At least one of them, say ω , must occur infinitely many times in a subsequence $\{D_{\gamma(x_{k_i})}\}_i$ of $\{D_{\gamma(x_k)}\}_k$. For each $j \in \text{sup}(\omega)$, consider the sequence of points $\{y_{j,i} \in \gamma(x_{k_i})\}_i$ which occupy the j -th position in $\gamma(x_{k_i}) \cap \partial_1 X$. By its definition, $y_{j,i} \in \partial_{\omega(j)} X^\circ$. Since $\partial_{\omega(j)} X$ is compact, there exists a subsequence $\{y_{j,i_l}\}_l$ that converges to a point $y_j \in \partial_{\omega(j)} X$. Its multiplicity, $m(y_j)$, is $\omega(j)$ at least.

Consider the segments $\gamma_{j,j+1;i}$ of $\gamma(x_{k_i})$ bounded by the adjacent pairs $(y_{j,i}, y_{j+1,i})$. The interiors of the segments do not intersect the boundary $\partial_1 X$. By the continuous dependence of v -integral curves on their end points, $\{\gamma_{j,j+1;i_l}\}_l$ converge to a trajectory segment $\gamma_{j,j+1}$ that connects y_j with y_{j+1} (that segment may contain "new" points from $\partial_1 X$ in its interior). As a result, all the y_j 's belong to the same trajectory γ . Since each $x_{k_{i_l}}$ belongs to some segment $\gamma_{j,j+1;i_l}$ and $\lim_{l \rightarrow +\infty} x_{k_{i_l}} = x$, we get $\gamma(x) = \gamma$.

By the argument above, $\text{sup}(D_{\gamma(x)})$ may contain new points that are not in the limit of $\cup_l \text{sup}(D_{\gamma(x_{k_{i_l}})})$. This can only boost the reduced multiplicity $|D_{\gamma(x)}|'$ in comparison

to $|D_{\gamma(x_{k_{i_l}})}|'$. On the other hand, some points in $\sup(D_{\gamma(x)})$ are the result of merging of two or several consecutive points $\{y_{j,i_l}\}_j$ from $\gamma(x_{k_{i_l}})$. In the process, thanks to the local models (1.5), their multiplicities add; so again, the reduced multiplicity of the limiting point exceeds the sum of the reduced multiplicities of the corresponding merging points. Therefore,

$$m'(\gamma(x)) \geq \overline{\lim}_{k \rightarrow +\infty} m'(\gamma(x_k)).$$

Moreover, if the combinatorial type ω of $\gamma(x_{k_i}) \cap \partial_1 X$ differs from the combinatorial type ω' of $\gamma(x) \cap \partial_1 X$, then $m'(\gamma(x)) > \overline{\lim}_{k \rightarrow +\infty} m'(\gamma(x_k))$. Similarly, $m(\gamma(x)) \geq \overline{\lim}_{k \rightarrow +\infty} m(\gamma(x_k))$.

The same argument shows that the relation between the types ω and ω' is exactly the order relation $\omega \succ_{\bullet} \omega'$ in Ω^{\bullet} , as introduced in Definition 2.5. \square

3. ON STRATIFIED SPACES

Let (\mathcal{S}, \succ) be a poset (a partially ordered set). Given a subset $\Theta \subset \mathcal{S}$ in the poset (\mathcal{S}, \succ) , we denote by Θ_{\succeq} the set

$$\{a \in \mathcal{S} \mid b \succeq a \text{ for some } b \in \Theta\}.$$

Let $\Theta_{\succ} := \Theta_{\succeq} \setminus \Theta$. Similarly, we introduce

$$\Theta_{\preceq} := \{a \in \mathcal{S} \mid a \succeq b \text{ for some } b \in \Theta\} \text{ and } \Theta_{\prec} := \Theta_{\preceq} \setminus \Theta.$$

In particular, for any $\omega \in \Omega^{\bullet}$, we will consider routinely the finite posets $\omega_{\succeq \bullet}$, $\omega_{\succ \bullet}$, and, for any $\omega \in \Omega$, the finite posets ω_{\succeq} , ω_{\succ} .

Definition 3.1. A \mathcal{S} -filtration of a topological space X is a collection of closed topological subspaces $\{X_{\preceq \omega}\}_{\omega \in \mathcal{S}}$ such that $\omega \prec \omega'$ implies the inclusion $X_{\preceq \omega} \subset X_{\preceq \omega'}$. Moreover, we require that each point $x \in X$ which belongs to some stratum $X_{\preceq \omega}$, $\omega \in \mathcal{S}$, belongs to a unique smallest stratum $X_{\preceq \omega(x)}$ labelled by a minimal element $\omega(x)$ from the poset $\{\preceq \omega\} \subset \mathcal{S}$. \square

When X itself is a member of the collection $\{X_{\preceq \omega}\}_{\omega \in \mathcal{S}}$, we assume that \mathcal{S} has a unique maximal element ω_* and that $X_{\preceq \omega_*} = X$.

For an \mathcal{S} -filtered space X , we define

$$X_{\prec \omega} := \bigcup_{\omega' \prec \omega} X_{\preceq \omega'} \text{ and } X_{\omega} := X_{\preceq \omega} \setminus X_{\prec \omega}.$$

In general, for any subset $\Theta \subset \mathcal{S}$, put $X_{\preceq \Theta} := \bigcup_{\omega \in \Theta} X_{\preceq \omega}$.

Definition 3.2. An \mathcal{S} -filtered map $f : X \rightarrow Y$ between \mathcal{S} -filtered spaces X and Y is a continuous map such that $f(X_{\preceq \omega}) \subset Y_{\preceq \omega}$ for each $\omega \in \mathcal{S}$.

A \mathcal{S} -filtered homotopy between \mathcal{S} -filtered maps $f_0 : X \rightarrow Y$ and $f_1 : X \rightarrow Y$ is an \mathcal{S} -filtered map $F : X \times [0, 1] \rightarrow Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in X$. \square

4. SPACES OF REAL POLYNOMIALS, STRATIFIED BY THE COMBINATORIAL TYPES OF THEIR REAL DIVISORS

Next, we would like to investigate carefully one natural stratification in the coefficient space $\mathbb{R}_{\text{coef}}^d$ of real monic polynomials $P(z)$ of a given degree d . This stratification is generated by different combinatorial patterns of zero divisors $D_{\mathbb{R}}(P)$ in \mathbb{R} . Its importance for our program is justified by the local models for transversally generic fields that have been described in Lemmas 1.1 and 1.2.

Let $\tau : \mathbb{C} \rightarrow \mathbb{C}$ be the complex conjugation. Via the Viète Map V , the coefficient space $\mathbb{R}_{\text{coef}}^d$ can be identified with the space $(\text{Sym}^d \mathbb{C})^\tau$ of τ -invariant divisors in \mathbb{C} of degree d .

Each function $\omega : \mathbb{N} \rightarrow \mathbb{Z}_+$ of l_1 -norm $|\omega| \leq d$ and such that $|\omega| \equiv d \pmod{2}$ defines a pure stratum $(\text{Sym}^d \mathbb{C})_\omega^\tau$ in $(\text{Sym}^d \mathbb{C})^\tau$ and therefore, in $\mathbb{R}_{\text{coef}}^d$. We denote this stratum $V(\text{Sym}^d \mathbb{C})_\omega^\tau$ by R_ω . In $\mathbb{R}_{\text{coef}}^d$, it represents all monic real polynomial P such that the combinatorics of $D_{\mathbb{R}}(P)$ is prescribed by ω . We denote by $(\text{Sym}^d \mathbb{C})_{\omega_{\geq}}^\tau$ the closure of the stratum $[(\text{Sym}^d \mathbb{C})_\omega]^\tau$, and by $R_{\omega_{\geq}}$ the closure of the stratum R_ω , respectively.

The depressed form of the polynomial in (1.5) and (1.7) calls for the introduction of similar spaces built out of, so called, *balanced divisors*.

Let $\alpha \in \mathbb{R}$. A divisor $D = \oplus_i m(i)z_i$ with $z_i \in \mathbb{C}$ is called α -balanced, if

$$\sum_i m(i)z_i = \alpha \cdot \sum_i m(i)$$

in \mathbb{C} . In other words, α is the center of gravity of the configuration of points-particles in \mathbb{C} representing D . In the root space $\text{Sym}^d \mathbb{C} = \{z_1, \dots, z_d\}$ (where $d = |\omega|$), such divisors are characterized by the equation $\sum_{k=1}^d z_k = \alpha \cdot d$.

The conjugation-invariant α -balanced divisors from $(\text{Sym}^d(\mathbb{C}))^\tau$ are described by the equation

$$\sum_{k=1}^d \text{Re}(z_k) = d \cdot \alpha.$$

They form a real hypersurface $(\text{Sym}_\alpha^d \mathbb{C})^\tau$ in $(\text{Sym}^d \mathbb{C})^\tau$. Zero-balanced divisors (i.e., $\alpha = 0$) are simply called *balanced*.

The image of $(\text{Sym}_\alpha^d \mathbb{C})^\tau$ under the Viète map V consists of real monic polynomials whose z^{d-1} -coefficient is α . In a similar way, we introduce the stratification

$$\{R_{\alpha, \omega} := V((\text{Sym}_\alpha^d \mathbb{C})_\omega^\tau)\}_\omega$$

in the $(d-1)$ -dimensional affine space $\mathbb{R}_{\text{coef}, \alpha}^d$ of such polynomials.

Let us stress that here we use ω 's that attach multiplicities only to *real* roots (their norms $|\omega|$ do not exceed d and $|\omega| \equiv d \pmod{2}$)!

Consider the upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} \mid \text{Re}(z) \geq 0\}$$

, and let

$$\mathbb{H}^\circ := \{z \in \mathbb{C} \mid \text{Re}(z) > 0\}.$$

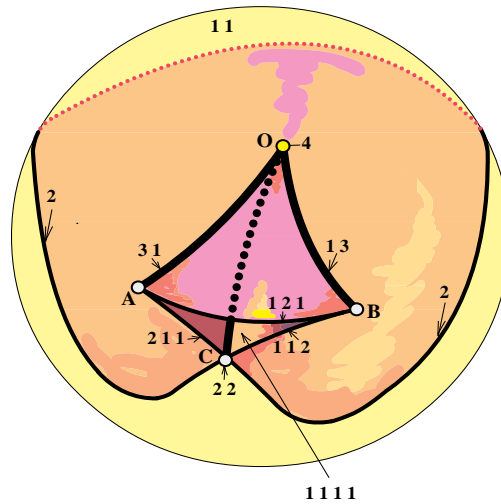


FIGURE 3. The Swallow Tail singularity is linked to the Whitney projection of the hypersurface $\{z^4 + x_2z^2 + x_1z + x_0 = 0\}$ onto the space $\mathbb{R}_{\text{coef},0}^3$ with the coordinates (x_0, x_1, x_2) . The strata in $\mathbb{R}_{\text{coef},0}^3$ are indexed by elements $\omega \in \Omega_{[4]}$. They divide the target space into three 3-cells, four 2-cells, three 1-cells, and one 0-cell.

Let $B^2 \subset \mathbb{C}$ be the unit ball centered on the origin. Put $B_+^2 = B^2 \cap \mathbb{H}$. Similarly, for any $\alpha \in \mathbb{R}$, let $B^2(\alpha) \subset \mathbb{C}$ be the unit ball centered on α , and $B_+^2(\alpha) := B^2(\alpha) \cap \mathbb{H}$.

Lemma 4.1. *The symmetric product $\text{Sym}^m \mathbb{H}$ is homeomorphic to the half-space \mathbb{R}_+^{2m} , bounded by a hyperplane in \mathbb{R}^{2m} .*

Let B^2 denotes a closed 2-ball. The symmetric product $\text{Sym}^m B^2$ is homeomorphic to a closed $2m$ -ball B^{2m} .

Proof. First we validate the second claim of the lemma.

The origin-centered dilation $t : \mathbb{C} \rightarrow \mathbb{C}$ induces an action ψ_t on the “root space” $\text{Sym}^m \mathbb{C}$: under this action, the support of each divisor is scaled by the factor $t > 0$. With the help of the Viète homeomorphism $V : \text{Sym}^m \mathbb{C} \rightarrow \mathbb{C}_{\text{coef}}^m$, this ψ_t -action induces a Ψ_t -action on the space $\mathbb{C}_{\text{coef}}^m$.

The boundary of $\text{Sym}^m B^2$ consists of divisors D whose support is contained in B^2 and has a non-empty intersection with the circle ∂B^2 . We notice that each ψ_t -trajectory through a point $D \in \text{Sym}^m B^2$, distinct from the divisor $m\{0\}$, has a unique point of intersection with the boundary $\partial(\text{Sym}^m B^2)$. Indeed, for each configuration $\Delta \neq \{0\}$ of points in B^2 , there is a unique origin-centered dilation $t : \mathbb{C} \rightarrow \mathbb{C}$ such that $t(\Delta) \cap \partial B^2 \neq \emptyset$ and $t(\Delta) \subset B^2$. Since the Viète map $V : \text{Sym}^m \mathbb{C} \rightarrow \mathbb{C}_{\text{coef}}^m$ is a smooth homeomorphism, $V(\text{Sym}^m B^2) \subset \mathbb{C}_{\text{coef}}^m$ is homeomorphic to $\text{Sym}^m B^2$. Therefore each trajectory Ψ_t -trajectory, except for the trivial trajectory through $m\{0\}$, hits $\partial V(\text{Sym}^m B^2) = V(\partial(\text{Sym}^m B^2))$ transversally at a singleton. The same property is shared by any $(2m - 1)$ -sphere in $\mathbb{C}_{\text{coef}}^m$ that is centered on the origin

0. Therefore, with the help of Ψ_t , $\partial V(\text{Sym}^m B^2)$ is homeomorphic to a $(2m - 1)$ -sphere, and $V(\text{Sym}^m B^2)$ to a closed ball B^{2m} . Hence $\text{Sym}^m B^2 \approx B^{2m}$ topologically.

A similar argument helps to analyze the topology of $\text{Sym}^m(\mathbb{H})$.

The boundary of the closed half-ball $B_+^2 := B^2 \cap \mathbb{H}$ consists of the segment $I = [-1, 1] \subset \mathbb{R}$ and the arc $A := \{z \in \mathbb{C} \mid |z| = 1, \text{Im}(z) \geq 0\}$.

Consider the set \mathcal{S} of divisors $D \in \text{Sym}^m \mathbb{H}$ such that their support $\Delta \subset B_+^2$ and $\Delta \cap A \neq \emptyset$. Again, there is a unique origin-centered dilation $t : \mathbb{C} \rightarrow \mathbb{C}$ such that $t(\Delta) \cap A \neq \emptyset$ and $t(\Delta) \subset B_+^2$, provided that $D \neq m\{0\}$. Therefore each ψ_t -trajectory through a point $D \in \text{Sym}^m \mathbb{H}$, except for the trivial trajectory through $m\{0\}$, has a unique point of intersection with \mathcal{S} .

The boundary $\partial(\text{Sym}^m \mathbb{H})$ in $\text{Sym}^m \mathbb{C}$ consists of divisors D such that their support Δ has a nonempty intersection with the real line $\mathbb{R} \subset \mathbb{H}$. Evidently, $\partial(\text{Sym}^m \mathbb{H})$ is invariant under ψ_t -flow. Thus, $\partial(\text{Sym}^m \mathbb{H})$ topologically is a cone over $\partial(\text{Sym}^m \mathbb{H}) \cap \mathcal{S}$, the set of divisors whose support $\Delta \subset B_+^2$ and such that $\Delta \cap I \neq \emptyset$.

Now, take a point $p \in \partial B^2$ and a small *open* ϵ -ball $B_\epsilon(p) \subset \mathbb{C}$ with the center at p . Since $H := B_\epsilon(p) \cap B^2$ is homeomorphic to the half-plane \mathbb{H} , $\text{Sym}^m \mathbb{H} \approx \text{Sym}^m H$.

Consider $m \cdot p$ as a point in $\partial(\text{Sym}^m B^2) \approx \partial B^{2m}$. Then $\text{Sym}^m H$ can be viewed as an open ϵ -neighborhood of $m \cdot p$ in the space $\text{Sym}^m B^2 \approx B^{2m}$, the distance in the closed ball $\text{Sym}^m B^2$ being the Hausdorff distance between S_m -orbits in $(B^2)^m \subset \mathbb{C}^m$. At least for a sufficiently small $\epsilon > 0$, that neighborhood $\text{Sym}^m H$, being an ϵ -neighborhood of a point $m \cdot p \in \partial(B^{2m})$, is homeomorphic to \mathbb{R}_+^{2m} , and so is $\text{Sym}^m \mathbb{H}$. \square

For $\omega \in \Omega_{\langle d \rangle}$, consider the space

$$(4.1) \quad e_\omega := \text{Sym}^{|\text{sup}(\omega)|} \mathbb{R} \times \text{Sym}^{\frac{d-|\omega|}{2}} \mathbb{H}.$$

Denote by σ_ω the subset of e_ω that consists of divisor pairs $D' \times D''$ such that $\text{sup}(D') \cup \text{sup}(D'') \subset B^2$, but not of its interior.

Let the subset $e_{\alpha, \omega} \subset e_\omega$ be defined by the constraint: the divisor $D' + D'' + \tau(D'')$ is α -balanced. Similarly, let $\sigma_{\alpha, \omega} \subset \sigma_\omega$ be defined by the property of $D' + D'' + \tau(D'')$ being α -balanced.

In what follows, by a “cell complex” we mean a Hausdorff topological space Z which is a disjoint union of its subsets $\{e_\alpha^\circ\}_\alpha$, where each e_α° is homeomorphic to an open ball B_α° of some dimension d_α . Let B_α be either a closed ball, or an open ball together with a northern hemisphere in its spherical boundary⁶. The homeomorphism $B_\alpha^\circ \rightarrow e_\alpha^\circ$ must extend to a continuous map $\phi_\alpha : B_\alpha \rightarrow Z$, so that the image $\phi_\alpha(\partial B_\alpha)$ is contained in a union of finitely many cells e_β of dimensions lower than d_α . By definition, $Y \subset Z$ is closed if, for each α , $\phi_\alpha^{-1}(Y \cap \phi_\alpha(B_\alpha))$ is closed in B_α . Note that e_α , the closure of e_α° in Z , coincides with $\phi_\alpha(B_\alpha)$. We do not require that e_α will be homeomorphic to a closed ball.

⁶This deviation from the standard definition of *CW*-complex is due to our need to consider germs of classical *CW*-complexes.

The next proposition describes one particularly important cellular structure on the space $\mathcal{P}^d \approx \mathbb{R}_{\text{coef}}^d$ of real degree d monic polynomials, the structure induced by the combinatorial types ω of their real divisors.

Theorem 4.1. *Let $\omega \in \Omega_{[d]}$. Then the following structures are available:*

- *each pure stratum $R_\omega \subset \mathbb{R}_{\text{coef}}^d$ is homeomorphic to an open ball⁷ of codimension $|\omega|'$.*
- *the space σ_ω is homeomorphic to a closed $(d - |\omega|' - 1)$ -ball, and the space $\sigma_{\alpha, \omega}$ to a closed $(d - |\omega|' - 2)$ -ball. The space e_ω is a positive cone over σ_ω , and $e_{\alpha, \omega}$ is a positive cone over $\sigma_{\alpha, \omega}$.*
- *for each $\hat{\omega} \in \Omega_{[d]}$, the strata $\{R_\omega\}_{\omega \preceq \hat{\omega}}$, define the structure of a cell complex on the real affine variety $R_{\hat{\omega} \succeq} \subset \mathbb{R}_{\text{coef}}^d$. The attaching maps*

$$\Phi_\omega^\partial : \partial e_\omega \rightarrow R_{\omega \succeq} \setminus R_\omega$$

for the cells e_ω are described in formulas (4.4)⁸.

- *the space $\mathbb{R}_{\text{coef}}^d$ admits a 1-parameter flow Ψ_t which has a single stationary point $\mathbf{0}$ (a source), is transversal to each sphere $S_{\text{coef}}^{d-1} \subset \mathbb{R}_{\text{coef}}^d$, centered on $\mathbf{0}$, and preserves each stratum R_ω . Thus, the cellular structure on $\mathbb{R}_{\text{coef}}^d$ (described in the third bullet) is a cone over a similar cellular structure on S_{coef}^{d-1} .*

Proof. Any divisor in \mathbb{R} comes with a particular linear order among the points in its support. Let P be a real polynomial of degree d . Its real divisor $D_{\mathbb{R}}(P)$ of the combinatorial type ω is determined (with the help of ω) by the support $\text{sup}(D_{\mathbb{R}}(P))$, a point the chamber $\Pi_{\mathbf{o}}^p$ of $\mathbb{R}^p = \{(y_1, \dots, y_p)\}$, defined by the inequalities $y_1 < y_2 < \dots < y_p$, where $p = |\text{sup}(\omega)|$. In fact, $\Pi_{\mathbf{o}}^p$ is one of the 2^{p-1} chambers-cells in which the hyperplanes $\{y_i = y_{i+1}\}_i$ divide \mathbb{R}^p .

The set $\Pi_{\mathbf{o}}^p$, admits an obvious embedding into the space $\mathbb{R}^{|\omega|}$ with the coordinates $(x_1, \dots, x_{|\omega|})$. It is defined there by the system of ω -dependent equations and inequalities:

$$x_1 = x_2 = \dots = x_{\omega(1)} < x_{\omega(1)+1} = x_{\omega(1)+2} = \dots = x_{\omega(1)+\omega(2)} < \dots$$

We denote by $\Pi_\omega^{\mathbf{o}}$ the solution set of this system. Note that $\Pi_\omega^{\mathbf{o}}$ is homeomorphic to the open chamber $\Pi_{\mathbf{o}}^{|\text{sup}(\omega)|} \subset \mathbb{R}^{|\text{sup}(\omega)|}$. Let Π_ω be the closure of $\Pi_\omega^{\mathbf{o}}$ in $\mathbb{R}^{|\omega|}$. That closure is homeomorphic to $\Pi^{|\text{sup}(\omega)|} \approx \text{Sym}^{|\text{sup}(\omega)|} \mathbb{R}$. Indeed, each orbit of the natural action of the symmetric group $S_{|\text{sup}(\omega)|}$ on $\mathbb{R}^{|\text{sup}(\omega)|}$ intersects the closed chamber $\Pi^{|\text{sup}(\omega)|}$ at a singleton.

With its real roots being fixed, a polynomial P is determined by the unordered pairs of its conjugate roots (possibly with multiplicities). Then the conjugate (non-real) pairs can be identified with points of the space $\text{Sym}^m \mathbb{H}^{\mathbf{o}}$, where $m = \frac{1}{2}(d - |\omega|)$. Therefore, $(\text{Sym}^d \mathbb{C})_\omega^\tau$ is homeomorphic to the space $\Pi_{\mathbf{o}}^p \times \text{Sym}^m \mathbb{H}^{\mathbf{o}}$, where $p := |\text{sup}(\omega)|$. Since $\mathbb{H}^{\mathbf{o}}$ is

⁷In general, the intersection of R_ω with a ball $B^d \subset \mathbb{R}^d$, centered at the origin, topologically is not a ball.

⁸See the proof of this theorem for the constructions of the relevant maps.

homeomorphic to \mathbb{C} and $\text{Sym}^m \mathbb{C}$ is homeomorphic to \mathbb{C}^m via the Viète Map V , we conclude that $\text{Sym}^m \mathbb{H}^\circ$ is homeomorphic to \mathbb{C}^m . Thus,

$$(\text{Sym}^d \mathbb{C})_\omega^\tau \approx \Pi_\omega^p \times \text{Sym}^m \mathbb{H}^\circ \approx \Pi_\omega^p \times \mathbb{C}^m$$

, an open cell of dimension

$$p + 2m = |\sup(\omega)| + (d - |\omega|) = d - |\omega|'.$$

Note that, the Viète map V generates a smooth homeomorphism between the spaces $(\text{Sym}^d \mathbb{C})_\omega^\tau \approx \mathbb{R}^{d-|\omega|'}$ and $R_\omega := (\mathbb{R}_{\text{coef}}^d)_\omega$. Thus the claim of the first bullet is validated.

The flow $\Psi_t : \mathbb{R}_{\text{coef}}^d \rightarrow \mathbb{R}_{\text{coef}}^d$ (see the fourth bullet) is induced by the flow ψ_t in the root space $(\text{Sym}^d \mathbb{C})^\tau$ that applies dilatations by real factors $t > 0$ to each τ -symmetric root configuration in \mathbb{C} . The transplantation of ψ_t to $\mathbb{R}_{\text{coef}}^d$ is done via the Viète Map V , a smooth homeomorphism $(\text{Sym}^d \mathbb{C})^\tau \rightarrow \mathbb{R}_{\text{coef}}^d$. Evidently, each stratum $(\text{Sym}^d \mathbb{C})_\omega^\tau$ is ψ_t -invariant; therefore each stratum R_ω must be invariant under Ψ_t .

We leave to the reader to verify that, for any polynomial $P \neq z^d$, its Ψ_t -trajectory is transversal to the spheres $S_{\text{coef}}^{d-1} \subset \mathbb{R}_{\text{coef}}^d$, centered on $\mathbf{0}$ (the verification is a straightforward computation). This proves the claim in the forth bullet.

Next, we turn our attention to the second bullet of the theorem. Each divisor $D \in \text{Sym}^m \mathbb{H}$ produces to a unique divisor $D_\mathbb{R}$, the part of D that is supported in \mathbb{R} . We view $D_\mathbb{R}$ as an element of $\text{Sym}^l \mathbb{R}$, $l \leq m$. For the majority of D 's, $D_\mathbb{R}$ will have an empty support, so we interpret $\text{Sym}^0 \mathbb{R}$ as an empty set.

Recall that $e_\omega := \Pi_\omega \times \text{Sym}^m \mathbb{H}$, where $m = \frac{1}{2}(d - |\omega|)$ (see (4.1)). In the proof of Lemma 4.1, we have seen that e_ω is homeomorphic to $\Pi_\omega \times \mathbb{R}_+^{d-|\omega|}$ —a $|\sup(\omega)|$ -dimensional pyramid times a half-space in $\mathbb{R}^{d-|\omega|}$.

The boundary of the unit half-disk $B_+^2 := \{z \in \mathbb{C} : |z| \leq 1, \text{Im}(z) \geq 0\}$ consists of the segment $I = [-1, 1] \subset \mathbb{R}$ and the arc

$$A := \{z \in \mathbb{C} : |z| = 1, \text{Im}(z) \geq 0\}.$$

The multiplicative group \mathbb{R}_+^* of positive real numbers acts semi-freely on the space $\Pi_\omega \times \text{Sym}^m \mathbb{H}$ by the diagonal ψ_t -action which is induced by the origin-centered dilatations in \mathbb{C} . Its only fixed point is the origin $\mathbf{0} := \{0\} \times \{0\}$. This \mathbb{R}_+^* -action on $e_\omega \setminus \mathbf{0}$ admits a compact section σ_ω that consists of points $D' \times D'' \in e_\omega$ such that $\sup(D') \cup \sup(D'')$ is contained in the unit half-disk $B_+^2 \subset \mathbb{H}$ (centered on the origin) and has a nonempty intersection with the arc A .

Thus σ_ω is the set of pairs $D' \in \text{Sym}^p(I)$, $D'' \in \text{Sym}^m(B_+^2)$ such that either $\sup(D') \cap \partial I \neq \emptyset$, or $\sup(D'') \cap A \neq \emptyset$, or both. Here $p = |\sup(\omega)|$. Note that if $\sup(D') \cap \partial I \neq \emptyset$, then $\sup(D') \cap A \neq \emptyset$ since $\partial A = \partial I$. Therefore, σ_ω can be also described as a set of pairs (D', D'') such that $\sup(D' + D'') \subset B_+^2$ and $\sup(D' + D'') \cap A \neq \emptyset$.

Recall that $\text{Sym}^p I \approx \Delta^p$, a p -simplex. By Lemma 4.1, $\text{Sym}^m B_+^2 \approx B_+^{2m}$, a half-ball. Let $\delta B_+^{2m} \subset \partial(B_+^{2m})$ denote the northern hemisphere in the boundary of the ball $B^{2m} \supset B_+^{2m}$. It corresponds to the divisors D'' with the property $\sup(D'') \cap A \neq \emptyset$.

In new notations, the section σ_ω is the set of pairs $D' \in \Delta^p, D'' \in B_+^{2m}$ such that either $D' \in \partial\Delta^p$, or $D'' \in \delta B_+^{2m}$, or both. As a result, we get a homeomorphism

$$\sigma_\omega \approx (\partial\Delta^p \times B_+^{2m}) \cup_{\partial\Delta^p \times \delta B_+^{2m}} (\Delta^p \times \delta B_+^{2m})$$

whose target topologically is a $(d - |\omega|' - 1)$ -ball. Indeed, as the formula above testifies, σ_ω is obtained from the solid torus $T := S^{p-1} \times D^{2m}$ by attaching a p -handle along $S^{p-1} \times D^{2m-1} \subset \partial T$.

Therefore e_ω , an infinite positive cone over the ball σ_ω , is homeomorphic to a positive cone in $\mathbb{R}^{d-|\omega|'}$ with a closed $(d - |\omega|' - 1)$ -ball base.

Now we proceed to describe the attaching maps (see (4.3) and (4.4)) for the cells e_ω . With this goal in mind, we will “partially compactify” the pure stratum R_ω in order to form an “honest” $(d - |\omega|')$ -cell e_ω and will show how to attach its boundary ∂e_ω to the strata $\{R_{\tilde{\omega}}\}_{\tilde{\omega} \in \omega_\succ}$ of dimensions smaller than $\dim(R_\omega)$. This cell e_ω can be regarded as a *resolution* of the real variety R_{ω_\succ} .

Consider the maps

$$(4.2) \quad \Theta_\omega : e_\omega := \Pi_\omega \times \text{Sym}^{\frac{d-|\omega|}{2}} \mathbb{H} \longrightarrow (\text{Sym}^d \mathbb{C})^\tau$$

defined by the formula $\Theta_\omega(D' \times D'') = D' + D'' + \tau(D'')$, where $D' \in \Pi_\omega$, $D'' \in \text{Sym}^{\frac{d-|\omega|}{2}} \mathbb{H}$, $\tau(D'')$ stands for the complex conjugate of the divisor D'' , and “+” denotes the sum of divisors in \mathbb{C} .

As formula (4.2) testifies, each map Θ_ω doubles the real part $D''_{\mathbb{R}}$ of each divisor $D'' \in \text{Sym}^m \mathbb{H}$ and thus mimics the merging of conjugate pairs of complex roots into the appropriate real roots of even multiplicity.

Note that the restriction of Θ_ω to $e_\omega^\circ := \Pi_\omega^\circ \times \text{Sym}^{\frac{d-|\omega|}{2}} \mathbb{H}^\circ$, the interior of e_ω , is a homeomorphism onto the pure stratum $(\text{Sym}^d \mathbb{C})_\omega^\tau$.

The restriction Θ_ω^∂ of Θ_ω to the boundary ∂e_ω provides us with the attaching maps

$$(4.3) \quad \{\Theta_\omega^\partial : \partial e_\omega \rightarrow (\text{Sym}^d \mathbb{C})^\tau\}_{\omega \in \Omega_{[d]}}.$$

By the very construction of e_ω , the Θ_ω -image of ∂e_ω belongs to the union of strata $\{(\text{Sym}^d \mathbb{C})_{\tilde{\omega}}^\tau\}_{\tilde{\omega} \in \omega_\succ}$, where $\tilde{\omega} \in \omega_\succ$. Indeed, if $D' \times D'' \in \partial e_\omega$, then either $D' \in \partial \Pi_\omega$, or $D'' \in \partial(\text{Sym}^{\frac{d-|\omega|}{2}} \mathbb{H})$. In the first case, D' is obtained from some $D \in \Pi_\omega^\circ$ via merge operations; thus $(D' + D'' + \tau(D''))_{\mathbb{R}}$ is obtained from $(D + D'' + \tau(D''))_{\mathbb{R}}$ by the same merges. In the second case, $(D' + D'' + \tau(D''))_{\mathbb{R}}$ can be obtained from D' by inserting $(D'' + \tau(D''))_{\mathbb{R}}$.

Therefore the maps $\{\Theta_\omega^\partial\}$ from (4.3) define cellular structures on the real affine variety $(\text{Sym}^d \mathbb{C})^\tau$ and its subvarieties $\{(\text{Sym}^d \mathbb{C})_{\tilde{\omega}_\succ}^\tau\}_{\tilde{\omega}_\succ \in \omega_\succ}$.

We notice that all the maps Θ_ω are equivariant under the \mathbb{R}_+^* -actions ψ_t and Ψ_t in the source and target spaces, respectively, so that the attaching maps are consistent with the cone structures of e_ω and of the strata $\{(\text{Sym}^d \mathbb{C})_{\tilde{\omega}}^\tau\}$. Moreover, the sections σ_ω are mapped by Θ_ω to some sections $S_\omega \subset R_\omega$ of the flow Ψ_t . The space S_ω is defined as the set of real polynomials with all their roots residing in the ball $B^2 \subset \mathbb{C}$, but not in its interior, and with the combinatorics of the real roots being prescribed by the poset $\omega_\succ \subset \Omega_{[d]}$.

With the help of the Viète homeomorphism V , the maps

$$(4.4) \quad \begin{aligned} \{\Phi_\omega : e_\omega \xrightarrow{\Theta_\omega} (\text{Sym}^d \mathbb{C})^\tau \xrightarrow{V} R_{\omega_\succeq} \subset \mathbb{R}_{\text{coef}}^d\}_{\omega \in \Omega_{\langle d \rangle}} \\ \{\Phi_\omega^\partial : \partial e_\omega \xrightarrow{\Theta_\omega^\partial} (\text{Sym}^d \mathbb{C})^\tau \xrightarrow{V} R_{\omega_\succ} \subset \mathbb{R}_{\text{coef}}^d\}_{\omega \in \Omega_{\langle d \rangle}} \end{aligned}$$

define cellular structures in $\mathbb{R}_{\text{coef}}^d$ and its subvarieties $\{R_{\hat{\omega}_\succeq}\}_{\hat{\omega}}$. Again, the attaching maps $\{\Phi_\omega, \Phi_\omega^\partial\}_\omega$ are \mathbb{R}_+^* -equivariant. They consistent with the cone structures in the strata $e_{\hat{\omega}}$ and $R_{\hat{\omega}}$. In particular, we get the maps:

$$(4.5) \quad \begin{aligned} \{\Phi_\omega : \sigma_\omega \xrightarrow{\Theta_\omega} (\text{Sym}^d \mathbb{C})^\tau \xrightarrow{V} S_{\omega_\succeq} \subset S_{\text{coef}}^{d-1}\}_{\omega \in \Omega_{\langle d \rangle}} \\ \{\Phi_\omega^\partial : \partial \sigma_\omega \xrightarrow{\Theta_\omega^\partial} (\text{Sym}^d \mathbb{C})^\tau \xrightarrow{V} S_{\omega_\succ} \subset S_{\text{coef}}^{d-1}\}_{\omega \in \Omega_{\langle d \rangle}} \end{aligned}$$

Here S_{coef}^{d-1} denotes the space of degree d real monic polynomials whose roots are in B^2 , but not in its interior. With the help of Ψ_t , the space S_{coef}^{d-1} is diffeomorphic to the standard sphere S^{d-1} . This completes the proof of the third bullet. \square

Results, similar to the ones described in Theorem 4.1, are valid for the space of real degree d monic polynomials with a fixed coefficient $d \cdot \alpha$ of z^{d-1} .

Theorem 4.2. *Let $\omega \in \Omega_{\langle d \rangle}$ and $\alpha \in \mathbb{R}$. We denote by $\omega_\star : 1 \rightarrow d$ the minimal element of the poset $\Omega_{\langle d \rangle}$. The following properties hold:*

- the stratum $R_{\alpha, \omega} \subset \mathbb{R}_{\text{coef}, \alpha}^{d-1}$ is an open ball of codimension $|\omega|'$,
- for each $\hat{\omega} \in \Omega_{\langle d \rangle}$, the strata $\{R_{\alpha, \omega}\}_{\omega \preceq \hat{\omega}}$ give rise to a cellular structure on the affine variety $R_{\alpha, \hat{\omega}_\succeq} \subset \mathbb{R}_{\text{coef}, \alpha}^{d-1}$. The attaching maps

$$\Phi_{\alpha, \omega}^\partial : \partial e_{\alpha, \omega} \rightarrow R_{\alpha, \omega_\succ} \setminus R_{\alpha, \omega}$$

for the cells $e_{\alpha, \omega}$ are described by formulas similar to formulas (4.4)-(4.5),

- the space $\mathbb{R}_{\text{coef}, \alpha}^{d-1}$ admits a 1-parameter flow Ψ_t^α which has a single stationary point O_α (a source), is transversal to each sphere $S_{\text{coef}, \alpha}^{d-2}$, centered on O_α , and preserves each stratum $R_{\alpha, \omega}$. Thus, the $\Omega_{\langle d \rangle}$ -labeled cellular structure on $\mathbb{R}_{\text{coef}, \alpha}^{d-1}$ is a cone over a similar $(\Omega_{\langle d \rangle} \setminus \omega_\star)$ -labelled cellular structure on the sphere $S_{\text{coef}, \alpha}^{d-2}$.

Proof. We turn to the α -balanced divisors, which tell a similar story.

Consider a flow $\phi_t : \{z_1, \dots, z_d\} \rightarrow \{z_1 - \frac{\alpha}{d}t, \dots, z_d - \frac{\alpha}{d}t\}$ in $(\text{Sym}^d \mathbb{C})^\tau$. For $t = 1$, it maps $\{z_1, \dots, z_d\}$ to $\{z_1 - \frac{\alpha}{d}, \dots, z_d - \frac{\alpha}{d}\}$, a α -balanced configuration. Note that ϕ_t preserves the ω -stratification in $(\text{Sym}^d \mathbb{C})^\tau$. Hence, with the help of the Viète map V , ϕ_t gives rise to a retraction of the stratum R_ω onto the stratum $R_{\alpha, \omega}$. Therefore, $R_{\alpha, \omega}$ is an open ball of dimension $d - |\omega|' - 1$ and codimension $|\omega|'$ in $\mathbb{R}_{\text{coef}, \alpha}^d$.

Next, consider the flow $\Psi_t^\alpha : \mathbb{R}_{\text{coef}, \alpha}^d \rightarrow \mathbb{R}_{\text{coef}, \alpha}^d$, induced with the help of the Viète map V by a flow $\{\psi_t^\alpha\}_{t>0}$ in the root space $(\text{Sym}_\alpha^d \mathbb{C})^\tau$. The flow ψ_t^α applies t -dilations that are centered on the point $o_\alpha := \alpha/d \in \mathbb{R}$ to each τ -symmetric and α -ballanced divisor D in \mathbb{C} . The dilatation ψ_t^α keeps the center of gravity of the weighted configuration $\psi_t^\alpha(D)$ at the

point $o_\alpha \in \mathbb{C}$, so that $\psi_t^\alpha(D)$ is α -balanced for all t . Evidently, these dilatations preserve the ω -type of each α -balanced configuration in \mathbb{C} . As a result, each $R_{\alpha,\omega}$ is Ψ_t^α -invariant.

By a direct computation, any Ψ_t^α -trajectory is transversal to the spheres $S_{\text{coef},\alpha}^{d-2} \subset \mathbb{R}_{\text{coef},\alpha}^d$ centered on the point $O_\alpha := V(o_\alpha, \dots, o_\alpha)$. The exception is the trivial trajectory through O_α .

We denote by $e_{\alpha,\omega}$ the subset of e_ω (see (4.1)) that consists of pairs (D', D'') , where $D' \in \Pi_\omega$, $D'' \in \text{Sym}^m \mathbb{H}$, and such that the divisor

$$D' + D'' + \tau(D'')$$

is α -balanced. In other words, if $D' = \sum_i m_i x_i$ and $D'' = \sum_k m_k z_k$, then $e_{\alpha,\omega}$ is the space of a fibration over the base $\text{Sym}^m \mathbb{H} \approx \mathbb{R}_+^{2m}$ with the cell-like fiber $F_{D''} \subset \Pi_\omega$ (over the point D'') that is defined by the constraint

$$\sum_i m_i x_i = d \cdot \alpha - \sum_k 2m_k \text{Re}(z_k).$$

Denote by $\sigma_{\alpha,\omega}$ the subset of $e_{\alpha,\omega}$ that consists of α -balanced pairs (D', D'') such that $\text{sup}(D') \cup \text{sup}(D'')$ is contained in the α -centered unit ball $B_+^2(\alpha)$, but not in its interior. As in the “unbalanced” case, $\sigma_{\alpha,\omega}$ is a section of the ψ_t^α -flow in $e_{\alpha,\omega}$. This results in $e_{\alpha,\omega}$ being a positive cone over the base $\sigma_{\alpha,\omega}$. By an argument as in the proof of Theorem 4.1, $\sigma_{\alpha,\omega}$ is homeomorphic to a closed $(d-2-|\omega'|)$ -ball.

For any real α , the maps Θ_ω in (4.2), being restricted to $e_{\alpha,\omega} \subset e_\omega$, produce the maps $\Theta_{\alpha,\omega} : e_{\alpha,\omega} \rightarrow (\text{Sym}_\alpha^d \mathbb{C})^\tau$ which, with the help of V , give rise to the maps

$$\Phi_{\alpha,\omega} : e_{\alpha,\omega} \rightarrow R_{\alpha,\omega_\succeq} \subset \mathbb{R}_{\text{coef},\alpha}^{d-1}$$

, already familiar from the “unbalanced” formulas (4.4). They define a cellular structure on the real variety $\mathbb{R}_{\text{coef},\alpha}^{d-1}$ and its subvarieties $R_{\alpha,\tilde{\omega}_\succeq}$. Analogously, the maps

$$\Phi_{\alpha,\omega} : \sigma_{\alpha,\omega} \rightarrow S_{\alpha,\omega_\succeq} \subset S_{\text{coef},\alpha}^{d-2}$$

define a cellular structure on the sphere $S_{\text{coef},\alpha}^{d-2}$ and its strata $S_{\alpha,\tilde{\omega}_\succeq}$. Here $S_{\text{coef},\alpha}^{d-2}$ is the space of real monic polynomials $P(z)$ of the form

$$z^d + \alpha z^{d-1} + \dots$$

such that the P -roots are contained in $B^2(\alpha)$, but not in its interior. Note that $P(z) \neq (z - \frac{\alpha}{d})^d$ —the apex of the cone. Similarly, $S_{\alpha,\omega_\succeq} \subset S_{\text{coef},\alpha}^{d-2}$, is the set of such polynomials $P(z)$ whose real roots have the combinatorics that is prescribed by the poset ω_\succeq . \square

Example 4.1. Let ω_\star be the minimal element of the poset $\Omega_{\langle d \rangle}$. Theorem 4.1 claims that the sphere S^{d-1} admits a cellular structure whose cells $\{S_\omega\}$ of codimension $|\omega'|$ are indexed by the elements of the poset $(\Omega_{\langle d \rangle} \setminus \omega_\star, \succ)$. Moreover, $S_{\omega'_\succeq} \subset S_{\omega''_\succeq}$ if and only if $\omega'' \succeq \omega'$. In other words, the poset $(\Omega_{\langle d \rangle} \setminus \omega_\star, \succ)$ provides a complete set of instructions for assembling S^{d-1} ! You may glance at Fig. 3 to examine how the assembly works for S^2 .

Note that this cellular structure is not a regular one: the closures $S_{\omega_{\succeq}}$ of cells S_{ω} are not necessarily closed balls. \square

The next lemma describes the attaching maps $\Phi_{\omega}^{\partial} : \partial e_{\omega} \rightarrow R_{\text{coef}}^d$ for the cells e_{ω} as being finitely ramified over their images. The degrees of ramification are described by combinatorics-flavored numbers $\{o(\omega, \tilde{\omega})\}$ whose exact Definition 4.1 will be provided below.

Lemma 4.2. *Each map $\Phi_{\omega} : e_{\omega} \rightarrow \mathbb{R}_{\text{coef}}^d$ from (4.4) has finite fibers, and is a bijection on the interior of the cell e_{ω} .*

For any point-polynomial $Q \in \Phi_{\omega}(\partial e_{\omega})$, the cardinality of the fiber $\Phi_{\omega}^{-1}(Q)$ is equal to the number $o(\omega, \tilde{\omega})$ introduced in Definition 4.1 below. Here $\tilde{\omega}$ denotes the combinatorial pattern of the real divisor $D_{\mathbb{R}}(Q)$.

Proof. Let us fix a degree d conjugation-invariant divisor

$$D = \Theta_{\omega}((D', D'')) := D' + D'' + \tau(D'')$$

, where $D' \in \Pi_{\omega}$, $D'' \in \text{Sym}^m \mathbb{H}$, and $m := (d - |\omega|)/2$. We denote by $\tilde{\omega}$ the combinatorial type of $D_{\mathbb{R}}$. Then there exist only finitely many pairs (D', D'') that deliver D . Indeed, since $\text{sup}(D') \subset \mathbb{R}$, the divisor $D'' - D''_{\mathbb{R}}$ with the support in \mathbb{H}° is uniquely determined by D . Note that $D_{\mathbb{R}}$ and D' (with the support in \mathbb{R}) differ by $2D''_{\mathbb{R}}$. This leaves only a finite set of choices for $D' = D_{\mathbb{R}} - 2D''_{\mathbb{R}}$, whose support must be contained in $\text{sup}(D_{\mathbb{R}})$ and whose degree is bounded by $\deg(D_{\mathbb{R}})$.

If $(D', D'') \in \partial e_{\omega}$, then either $D' \in \partial \Pi_{\omega}$, or $D'' \in \partial(\text{Sym}^m \mathbb{H})$ (i.e. $\text{sup}(D'') \cap \mathbb{R} \neq \emptyset$), or both. In the first case, $\omega(D')$, the combinatorial type of D' , is obtained from ω by a sequence of merge operations $\{M_j\}$. In the second case, $\tilde{\omega} := \omega(D_{\mathbb{R}})$, the combinatorial type of the Θ_{ω} -image of (D', D'') , is obtained from ω by a sequence of insert operations $\{I_k\}$. In the third mixed case, one applies both types of operations.

As a result, the combinatorics of reconstructing (D', D'') from $D = D' + D'' + \tau(D'')$ can be described as follows. If $\tilde{\omega}$ is the combinatorial pattern of $D_{\mathbb{R}}$, then ω' , the combinatorial pattern of $D' = D'_{\mathbb{R}}$ can be obtained by: (1) subtracting from $\tilde{\omega}$ a non-negative function $2\omega''$, such that $\tilde{\omega} - 2\omega''$ is again a non-negative function (ω'' represents the divisor $D''_{\mathbb{R}}$), and (2) deleting all the positions i where $(\tilde{\omega} - 2\omega'')(i) = 0$.

Let us denote by K this “repackaging” operator from (2); it takes any histogram and deletes from it all the columns of zero height.

Since $D' \in \partial \Pi_{\omega}$, we conclude that $\omega' = K(\tilde{\omega} - 2\omega'')$ should be obtainable from ω via a sequence of merge operations alone.

For each $\omega \in \Omega$, let us denote by $\omega_{\succeq M}$ the subset of Ω that consists of elements that can be obtained from ω by the merge operations $\{M_j\}$ alone.

In these notations, we get that $K(\tilde{\omega} - 2\omega'') \in \omega_{\succeq M}$.

Definition 4.1. *For any pair $\omega \succeq \tilde{\omega}$ in Ω , consider the subset $O(\omega, \tilde{\omega}) \subset \Omega$ such that, for any $\omega'' \in O(\omega, \tilde{\omega})$, the following properties hold:*

- the function $\tilde{\omega} - 2\omega'' \geq 0$
- $K(\tilde{\omega} - 2\omega'') \in \omega_{\succeq M}$

Let $o(\omega, \tilde{\omega})$ denote the cardinality of $O(\omega, \tilde{\omega})$. \square

Recall that the Viète map $V : (\text{Sym}^d \mathbb{C})^\tau \rightarrow \mathbb{R}_{\text{coef}}^d$ is a stratification-preserving homeomorphism of the Ω_d -stratified spaces; in particular, it is bijective. Therefore the cardinality of each fiber $\Phi_\omega^{-1}(Q)$, where $Q \in \mathbb{R}_{\text{coef}}^d$, is equal to the cardinality of the fiber $\Theta_\omega^{-1}(V^{-1}(Q))$.

Pick $Q \in \Phi_\omega(\partial e_\omega)$ and let $\tilde{\omega}$ denote the combinatorial type of the divisor $D_{\mathbb{R}}(Q)$. Then, by the considerations above, we have proved that $|\Phi_\omega^{-1}(Q)| = |\Theta_\omega^{-1}(V^{-1}(Q))| = o(\omega, \tilde{\omega})$. \square

For each $\omega \in \Omega_{\langle d \rangle}$, let $\omega_{\prec \rightsquigarrow k}$ be the set of elements $\tilde{\omega} \in \Omega_{\langle d \rangle}$ that can produce ω as a result of a sequence of k elementary operations $\{M_i, I_j\}$ being applied to $\tilde{\omega}$ (each such $\tilde{\omega}$ is the maximal element in a chain of ω -predecessors in $\Omega_{\langle d \rangle}$ of length k). Similar, let $\omega_{\preceq \rightsquigarrow k}$ be the set of elements in $\Omega_{\langle d \rangle}$ that can produce ω by a sequence of k elementary operations at most.

Next lemma is an extension of Theorem 4.1 and Theorem 4.2. It spells out the local arrangement of cells in the direction normal to a typical pure stratum R_ω or $R_{\alpha, \omega}$ in $\mathbb{R}_{\text{coef}}^d$ or in $\mathbb{R}_{\text{coef}, \alpha}^{d-1}$, respectively.

Lemma 4.3. *Let $\omega \in \Omega_{\langle d \rangle}$ and $P \in R_\omega$. We denote by $St_\nu(P)$ the star, normal in $\mathbb{R}_{\text{coef}}^d$ to the pure stratum R_ω at the point P . This normal star is homeomorphic to the space $\mathbb{R}^{|\omega|'}$ and inherits the structure of a cell complex from $\mathbb{R}_{\text{coef}}^d$. The cells $f_{\tilde{\omega}} := St_\nu(P) \cap R_{\tilde{\omega}}$ of dimension $|\omega|' - |\tilde{\omega}|'$ are indexed by the elements $\tilde{\omega} \succeq \omega$.*

The incidence of cells $\{f_{\tilde{\omega}}\}$ in $St_\nu(P)$ is prescribed by the partial order in the poset ω_{\preceq} : specifically, any element $\tilde{\omega} \in \omega_{\prec \rightsquigarrow k}$ gives rise to a cell $f_{\tilde{\omega}} \subset St_\nu(P)$ of dimension k . It is contained in exactly $|\tilde{\omega}|' + \#(\tilde{\omega}^{-1}(2))$ cells $\{f_{\tilde{\omega}'}\}$ of dimension $k+1$. The total number of cells $f_{\tilde{\omega}} \subset St_\nu(P)$ of dimension k is the cardinality of the set $\omega_{\prec \rightsquigarrow k}$.

Similar properties hold for the strata $\{St_\nu(P) \cap R_{\alpha, \tilde{\omega}}\}_{\tilde{\omega}}$ in the α -balanced polynomial space $\mathbb{R}_{\text{coef}, \alpha}^d$.

Proof. Let $St(R_\omega)$ be the union of all cells $\{R_{\tilde{\omega}}\}$ in $\mathbb{R}_{\text{coef}}^d$ such that $R_{\tilde{\omega}_\succeq} \supset R_\omega$. In contrast with the standard definition of the star, here we ignore the cells $R_{\tilde{\omega}}$ such that $R_{\tilde{\omega}_\succeq} \cap R_\omega \neq \emptyset$, but $R_{\tilde{\omega}_\succeq}$ does not contain R_ω .

Recall that any elementary operation M_j or I_j from (2.1)-(2.3), being applied to an element $\omega \in \Omega$, lowers its reduced norm $|\omega|'$ by 1. Thus each $R_{\tilde{\omega}}$ from $St(R_\omega)$ has $\text{codim}(R_\omega, R_{\tilde{\omega}_\succeq}) = k$, if and only if, $\tilde{\omega} \in \omega_{\prec \rightsquigarrow k}$, that is, if $\tilde{\omega}$ can be obtained from ω by a sequence of k elementary resolutions $\{M_j^{-1}\}^9$ and elementary reductions by two $\{I_j^{-1}\}^{10}$. In particular, there are exactly as many cells $R_{\tilde{\omega}}$ of dimension $\dim(R_\omega) + 1$ as there are elementary resolutions and reductions of ω . Each value $\omega(i)$ can be "resolved" in $\omega(i) - 1$ distinct ways:

$$(\omega(i) - 1, 1), (\omega(i) - 2, 2), \dots, (1, \omega(i) - 1)$$

(the order does matter!), and each $\omega(i) = 2$ can be "deleted" or "reduced". Therefore, the total number of elementary resolutions is $\sum_i (\omega(i) - 1) = |\omega|'$. The total number of

⁹These mimic a bifurcation of a real multiple root in a pair of real roots of the same combined multiplicity.

¹⁰These mimic the resolution of a real root α of multiplicity 2 into two simple complex-conjugate roots.

reductions is $\#(\omega^{-1}(2))$, the cardinality of $\omega^{-1}(2)$. All together, there are $|\omega|' + \#(\omega^{-1}(2))$ elementary operations applicable to ω . Stated differently, the *multiplicity* of each cell R_ω —the number of $(\dim(R_\omega) + 1)$ -cells containing it—is the *codimension* of R_ω in $\mathbb{R}_{\text{coef}}^d$ (or in $\mathbb{R}_{\text{coef},\alpha}^{d-1}$) plus $\#(\omega^{-1}(2))$.

By Theorem 4.1, $St_\nu(P)$, the germ of the vector space normal to R_ω at P , has dimension $|\omega|'$. It is transversal at P to each cell $R_{\tilde{\omega}}^-$ that contains R_ω . Thus, $St_\nu(P) \cap R_{\tilde{\omega}}^-$ is a germ-cell $f_{\tilde{\omega}}$ of dimension $|\omega|' - |\tilde{\omega}|'$. The intersection $R_{\tilde{\omega}} \cap L_\nu(P)$ with the normal link $L_\nu(P)$ —the boundary of $St_\nu(P)$ —is a cell $f_{\tilde{\omega}}^\partial$ of dimension $|\omega|' - |\tilde{\omega}|' - 1$. This reveals the cellular structure of $\mathbb{R}_{\text{coef}}^d$ and of $\mathbb{R}_{\text{coef},\alpha}^{d-1}$ at P as a product of the cellular structure in $St_\nu(P)$ times the open cell R_ω . Therefore, the same combinatorics $(\Omega_{\langle d \rangle}, \succ)$ governs both structures; in particular, each cell $f_{\tilde{\omega}} \subset St_\nu(P)$ is contained in exactly $|\tilde{\omega}|' + \#(\tilde{\omega}^{-1}(2))$ cells of the next dimension. Similarly, each cell $f_{\tilde{\omega}}^\partial \subset L_\nu(P)$, $\tilde{\omega} \neq \omega$, is contained in exactly $|\tilde{\omega}|' + \#(\tilde{\omega}^{-1}(2))$ cells of the next dimension. \square

Remark 4.1. Note that the combinatorics of these cellular structures is different from the combinatorics of the standard simplex Δ^d : in a simplex, the multiplicity of each subsimplex is equal to its codimension (to $|\tilde{\omega}|'$ in our notations); thus the “defect” $\#(\tilde{\omega}^{-1}(2))$ measures the deviation of the $\Omega_{\langle d \rangle}$ -labeled cellular structures in $\mathbb{R}_{\text{coef}}^d$ from the standard simplicial one in Δ^d . \square

5. ON SPACES OF MULTI-TANGENT TRAJECTORIES

As before, let X be a compact smooth $(n+1)$ -manifold with boundary. For a boundary generic field v , each v -trajectory γ intersects the boundary $\partial_1 X$ at a finite number of points a of multiplicities $m(a) \leq n+1$. Recall that formulas (1.2)-(1.4) attach the multiplicity $m(\gamma)$, the reduced multiplicity $m'(\gamma)$, and the virtual multiplicity $\mu(\gamma)$ to each trajectory γ . For transversally generic fields, $m(\gamma) \leq 2(n+1)$, $m'(\gamma) \leq n$ for every γ (see Theorem 3.4 from [K2]).

The space of v -trajectories $\mathcal{T}(v)$ is given the quotient topology, so that the obvious map $\Gamma : X \rightarrow \mathcal{T}(v)$ is continuous. Let us stress again that if v has singularities, the trajectory space $\mathcal{T}(v)$ is quite pathological (non-separable). In contrast, for nonsingular generic gradient fields $\mathcal{T}(v)$ is a decent space, a compact CW -complex! We will prove this theorem in the next paper.

For traversing fields v , all the fibers of Γ are closed intervals or singletons. This property leads to

Theorem 5.1. *For a transversally generic field v on X , the map $\Gamma : X \rightarrow \mathcal{T}(v)$ is a weak homotopy equivalence¹¹.*

For any traversing field v and any local coefficient system \mathcal{A} on X , the map Γ is a homology equivalence:

$$\Gamma^* : H^*(\mathcal{T}(v); \Gamma_*(\mathcal{A})) \approx H^*(X; \mathcal{A})$$

for all $* \geq 0$.

¹¹In the next paper, we will show that Γ actually is a homotopy equivalence for transversally generic fields.

Proof. Suppose there exists an open cover $\mathcal{U} = \{U_\alpha\}_\alpha$ of $\mathcal{T}(v)$, such that all the maps $\Gamma : \Gamma^{-1}(U_\alpha) \rightarrow U_\alpha$ are weak homotopy equivalences. Then, by Corollary 1.4 from [May], $\Gamma : X \rightarrow \mathcal{T}(v)$ is a weak homotopy equivalence, provided that the cover \mathcal{U} is closed under finite intersections of its elements.

So we need to construct the appropriate cover of $\mathcal{T}(v)$ and to prove that Γ is a weak homotopy equivalence just locally.

By Lemma 1.2, for each v -trajectory γ , there exists a \hat{v} -adjusted neighborhood $\hat{V}_\gamma \subset \hat{X}$ of γ such that, in special coordinates (u, x, y) , X is described by the polynomial inequality $\{P(u, x) \leq 0\}$, and the cylindrical neighborhood \hat{V}_γ of $\gamma \subset X$ by the additional inequalities $\|x\| \leq \epsilon$, $\|y\| \leq \epsilon'$.

Being the union of all \hat{v} -trajectories close to γ , this neighborhood \hat{V}_γ determines a neighborhood U_γ of the point $\gamma \in \mathcal{T}(v)$. Note that \hat{V}_γ may have \hat{v} -trajectories that do not intersect X . Let us denote by V_γ the subset of \hat{V}_γ built out of \hat{v} -trajectories $\hat{\gamma}$ with the property $\hat{\gamma} \cap X \neq \emptyset$. By definition, the restriction $\Gamma : V_\gamma \rightarrow U_\gamma$ is surjective.

Let us denote by $\omega_x \in \Omega$ the combinatorial pattern of the divisor $D_{\mathbb{R}}(P(u, x))$, where $x \in \mathbb{R}^{|\omega_0|'}$ and ω_0 is the combinatorial pattern of the divisor $D_{\mathbb{R}}(P(u, 0))$. If x is such that $P(u, x) > 0$ for all u , then $\omega_x : \mathbb{N}_+ \rightarrow \mathbb{Z}_+$ is defined to be the zero map. The reduced norm $|\omega_x|'$ of such trivial ω_x is defined to be -1 .

For each $k \in [0, |\omega_0|']$, consider the real subvariety $\mathcal{X}_k \subset \mathbb{R}^{|\omega_0|'}$ defined by the constraint $|\omega_x|' \geq k$. In other words, $x \in \mathcal{X}_k$ if and only if the reduced multiplicity $m'(D_{\mathbb{R}}(P(u, x))) \geq k$. In view of Theorem 4.1 and Theorem 4.2, $\text{codim}(\mathcal{X}_k, \mathbb{R}^{|\omega_0|'}) = k$.

Let $\pi : V_\gamma \rightarrow \mathbb{R}^{|\omega_0|'} \times \mathbb{R}^{n-|\omega_0|'}$ denote the projection $(u, x, y) \rightarrow (x, y)$.

Put

$$V_{\gamma,k} := \pi^{-1}(\mathcal{X}_k \times B_{\epsilon'}(0))$$

, where $B_{\epsilon'}(0)$ is the ϵ' -ball in $\mathbb{R}^{n-|\omega_0|'}$ with the center at 0. Thus $V_{\gamma,0} := V_\gamma$, and $V_{\gamma,|\omega_0|'} = \gamma \times B_{\epsilon'}(0)$.

The map π can be viewed as a composition $p \circ q$ of two maps: the quotient surjective map $q : V_\gamma \rightarrow U_\gamma \subset \mathcal{T}(v)$, whose fibers are closed segments, and a finitely ramified map $p : U_\gamma \rightarrow \mathbb{R}^{|\omega_0|'}$. The map q is the restriction of the map $\Gamma : X \rightarrow \mathcal{T}(v)$ to the neighborhood V_γ .

Let $U_{\gamma,k} := q(V_{\gamma,k})$. We will argue by induction “ $k \Rightarrow k-1$ ”. We claim that if $q : V_{\gamma,k} \cap X \rightarrow U_{\gamma,k}$ is a weak homotopy equivalence, then so is the map $q : V_{\gamma,k-1} \cap X \rightarrow U_{\gamma,k-1}$, provided $k > 0$.

First we will show that the map

$$\tilde{q}_{k-1} : (V_{\gamma,k-1} \cap X) / (V_{\gamma,k} \cap X) \rightarrow U_{\gamma,k-1} / U_{\gamma,k}$$

admits a continuous section σ_{k-1} such that $\sigma_{k-1}(U_{\gamma,k-1} / U_{\gamma,k})$ is a deformation retract of $(V_{\gamma,k-1} \cap X) / (V_{\gamma,k} \cap X)$.

For each $x \in \mathcal{X}_{k-1}$, the set $\{P(u, x) \leq 0\}$ is a disjointed union of closed intervals $\{I_i(x)\}_i$. Let $u_i^*(x)$ be the center of the interval $I_i(x)$. With the help of q , the pair $(x, u_i^*(x))$ determines the point $q(x, u_i^*(x))$ in $\mathcal{T}(v)$. Then we define the “protosection”

σ_{k-1} by the formula

$$\sigma_{k-1}(q(x, u_i^*(x))) := (x, u_i^*(x)).$$

This formula is discontinuous for points $q(x, u_i^*(x)) \in U_{\gamma,k}$, where some intervals

$$\{I_i(x)\}_{x \in \mathcal{X}_{k-1} \setminus \mathcal{X}_k, i}$$

merge; however, it produces a continuous section

$$\sigma_{k-1} : U_{\gamma,k-1}/U_{\gamma,k} \rightarrow (V_{\gamma,k-1} \cap X)/(V_{\gamma,k} \cap X)$$

of the quotients. Now $(V_{\gamma,k-1} \cap X)/(V_{\gamma,k} \cap X)$ retracts on $\sigma_{k-1}(U_{\gamma,k-1}/U_{\gamma,k})$ by collapsing each interval $I_i(x)$ on its center $u_i^*(x)$.

The basis of induction is $k = |\omega_0|'$. In this case, with the help of q , $V_{\gamma,|\omega_0|'} := \gamma \times B_{\epsilon'}(0)$ is homotopy equivalent to $U_{\gamma,|\omega_0|'} := B_{\epsilon'}(0)$.

By the inductive assumption, $q_k : V_{\gamma,k} \cap X \rightarrow U_{\gamma,k}$ is a weak homotopy equivalence. We have shown that

$$\tilde{q}_{k-1} : (V_{\gamma,k-1} \cap X)/(V_{\gamma,k} \cap X) \rightarrow U_{\gamma,k-1}/U_{\gamma,k}$$

is a homotopy equivalence. Therefore, comparing the exact homotopy sequences of the two triples, we conclude that $q_{k-1} : V_{\gamma,k-1} \cap X \rightarrow U_{\gamma,k-1}$ is a weak homotopy equivalence as well. In particular, it follows that $q_0 := q : V_{\gamma,0} \cap X \rightarrow U_{\gamma,0}$ is a weak homotopy equivalence.

By compactness of X , we can pick a finite v -adjusted cover $\mathcal{V} := \{V_\gamma\}$ of $X \subset \hat{X}$ and the corresponding cover $\mathcal{U} := \{U_\gamma := \Gamma(V_\gamma)\}$ of $\mathcal{T}(v)$, so that each map $\Gamma : V_\gamma \rightarrow U_\gamma$ is a weak homotopy equivalence. Add to the list \mathcal{V} all the multiple intersections $V_{\gamma_1} \cap V_{\gamma_2} \cap \cdots \cap V_{\gamma_r}$ of elements from \mathcal{V} , thus forming a larger lists $\hat{\mathcal{V}}$ and a new corresponding list $\hat{\mathcal{U}}$ comprising all the intersections $U_{\gamma_1} \cap U_{\gamma_2} \cap \cdots \cap U_{\gamma_r}$.

For each k , the locally-defined sets $\{V_{\gamma_l,k}\}_l$ have an intrinsic description in terms of the combinatorial patterns of tangency. So they automatically agree on multiple intersections: $X \cap V_{\gamma_l,k} \cap V_{\gamma_m,k} \subset X \cap V_{\gamma_{lm},k}$ for all l, m . Now the same inductive argument in k works for each map

$$\Gamma : X \cap V_{\gamma_1} \cap V_{\gamma_2} \cap \cdots \cap V_{\gamma_r} \rightarrow U_{\gamma_1} \cap U_{\gamma_2} \cap \cdots \cap U_{\gamma_r}$$

, so that this map is a weak homotopy equivalence as well.

As a result, by Corollary 1.4 [May], $\Gamma : X \rightarrow \mathcal{T}(v)$ is a weak homotopy equivalence.

Now consider a traversing field v on X . Let \mathcal{A} be any local coefficient system (a sheaf) on X with an abelian group A for the stock. We denote by $\Gamma_*(\mathcal{A})$ its push-forward residing on the trajectory space $\mathcal{T}(v)$.

Let U be an open neighborhood of a typical trajectory γ in X . Since X is compact and a typical Γ -fiber—a trajectory γ —is closed, the canonical homomorphism

$$\lim \operatorname{ind}_{\{U \supset \gamma\}} H^*(U; \mathcal{A}|_U) \rightarrow H^*(\gamma; \mathcal{A}|_\gamma)$$

is an isomorphism (see Theorem 4.11.1 in [God]). Since all γ 's are either segments or singletons, $H^*(\gamma; \mathcal{A}|_\gamma) = 0$ for all $* \neq 0$ and $H^0(\gamma; \mathcal{A}|_\gamma) = A$. Thus, the Leray spectral sequence

$$\{E_2^{pq} = H^p(\mathcal{T}(v); \mathcal{H}^q(\gamma; \mathcal{A}))\}_{p,q}$$

of the map $\Gamma : X \rightarrow \mathcal{T}(v)$ collapses (see Theorem 4.17.1 in [God]). As a result, we get that the map Γ establishes an isomorphism $\Gamma^* : H^*(\mathcal{T}(v); \Gamma_*(\mathcal{A})) \approx H^*(X; \mathcal{A})$.

In particular, for a trivial local system A , we get $H^*(X; A) \approx H^*(\mathcal{T}(v); A)$. \square

Remark 5.1. If a traversing field v is such that $\mathcal{T}(v)$ has a homotopy type of a CW -complex, then by Whitehead Theorem [Wh], $\Gamma : X \rightarrow \mathcal{T}(v)$ is a homotopy equivalence. In the next paper, we will prove that, for a transversally generic field v , the trajectory space $\mathcal{T}(v)$ can be given the structure of a compact CW -complex. \square

For any sub-poset $\Theta \subset \Omega_{\langle n \rangle}^\bullet$ and a transversally generic field v on X , let us consider the subsets $X(v, \Theta) \subset X$ and $\mathcal{T}(v, \Theta) \subset \mathcal{T}(v)$ comprised of points $x \in X$ or of trajectories $\gamma_x \in \mathcal{T}(v)$ whose divisors D_{γ_x} have the combinatorial models prescribed by the elements of Θ .

In particular, we will see that the webs of subspaces

$$\{X(v, \omega_{\succeq})\}_{\omega \in \Omega_{\langle n \rangle}^\bullet} \quad \text{and} \quad \{\mathcal{T}(v, \omega_{\succeq})\}_{\omega \in \Omega_{\langle n \rangle}^\bullet}$$

form a remarkable geometric structure. It will preoccupy us for the rest of this series of articles.

A cruder stratification (filtration) of X and $\mathcal{T}(v)$ is provided by the spaces

$$\{X(v, \Omega_{[k, n]}^\bullet)\}_{0 \leq k \leq n} \quad \text{and} \quad \{\mathcal{T}(v, \Omega_{[k, n]}^\bullet)\}_{0 \leq k \leq n}$$

, respectively.

Lemma 3.4 (see also Lemma 1.2) and Theorem 3.5 from [K2] have an useful implication:

Corollary 5.1. *Let X be a smooth $(n+1)$ -manifold with boundary. For any transversally generic field v , the obvious map $\Gamma : \partial_1 X \rightarrow \mathcal{T}(v)$ is $(n+2)$ -to-1 at most. At the same time, $\Gamma : \partial_2 X \rightarrow \mathcal{T}(v)$ is n -to-1 at most. For each ω , the restriction of Γ to the subspace $\Gamma^{-1}(\mathcal{T}(v, \omega))$ is $|\sup(\omega)|$ -to-1.*

When restricted to the Γ -preimage of the proper stratum $\mathcal{T}(v, \omega)$, Γ is a covering map with a trivial monodromy and a fiber of cardinality $|\sup(\omega)|$.

Proof. For a transversally generic field, by Theorem 3.5 from [K2], $m(\gamma) \leq 2(n+1)$. Each trajectory has exactly two points of odd multiplicity, the rest of the points are tangent points of even multiplicity. Their number does not exceed n . Thus Γ is $(n+2)$ -to-1 at most, and $\Gamma|_{\partial_2 X}$ is n -to-1 at most.

The statement dealing with the cardinalities of the fibers of

$$\Gamma : \Gamma^{-1}(\mathcal{T}(v, \omega)) \rightarrow \mathcal{T}(v, \omega)$$

follows instantly from the definitions of the relevant spaces.

Let β be a loop in $\mathcal{T}(v, \omega)$, and let $E_\beta := \Gamma^{-1}(\beta) \subset X$. Note that $\Gamma : E_\beta \rightarrow \beta$, thanks to the orientation by v , is a cylinder (and not a Möbius band). Consider the intersection $E_\beta \cap \partial_1 X$. Since β is contained in the pure stratum $\mathcal{T}(v, \omega)$, $\Gamma : E_\beta \cap \partial_1 X \rightarrow \beta$ is a covering map with a finite fiber. Because its space $E_\beta \cap \partial_1 X$ is contained in the cylinder E_β , we conclude that $\Gamma : E_\beta \cap \partial_1 X \rightarrow \beta$ must be a trivial covering. \square

Assuming that v is transversally generic, our immediate goal is to describe one particular *localized* cellular structure of the trajectory space $\mathcal{T}(v)$. As we mentioned before, it is governed by the combinatorics of the divisors in \mathbb{R} of real degree $\leq 2(n+1)$ polynomials.

First, we would like to understand better the $\Omega_{\langle n \rangle}^\bullet$ -stratified structure of $\mathcal{T}(v)$, localized to the vicinity of given v -trajectory γ .

As usual, we operate within an extension germ (\hat{X}, \hat{v}) of (X, v) . Let $\{a_i\}_i := \gamma \cap \partial_1 X$ be the v -ordered finite set of points, where $a_i \in \partial_{j_i} X^\circ$. By Theorem 3.5 from [K2] (see also Lemma 1.2), this tangency pattern $\omega = (j_1, j_2, \dots)$ is described by an element $\omega \in \Omega_{\langle n \rangle}^\bullet \cap \Omega_{\langle 2n+2 \rangle}$.

By Lemma 1.2 and formula (1.7), in special coordinates (u, x, y) on some \hat{v} -adjusted tube surrounding γ , the manifold X is given by the inequality

$$(5.1) \quad P(u, x) := \prod_i [(u - \alpha_i)^{j_i} + \sum_{l=0}^{j_i-2} x_{i,l} (u - \alpha_i)^l] \leq 0$$

, where $\alpha_i = u(a_i)$ and $x = \{x_{i,l}\}$.

Next, in our local analysis, we can pick a canonical model of X in the vicinity of γ by assuming that each $\alpha_i = i$.

If $|\omega| \equiv 0(2)$, for each fixed value of the coordinates (x, y) , the solution set of (5.1) is a disjoint union of several closed intervals and singletons residing in the u -line $\hat{\gamma}_x$ (see Fig. 1 in [K2]), the union depending on x alone. Each of these intervals and singletons represent a v -trajectory suspended over (x, y) (for some x , $\hat{\gamma}_x$ can be empty!). So to get the space of trajectories $\mathcal{T}(v)$ in the vicinity of γ , we need to collapse each interval to a point-marker that resides in it. Let us formalize the collapsing procedure.

Consider the solution set E_ω of (5.1). We say that two points $(u, x, y), (u', x', y') \in E_\omega$ are equivalent (" \sim "), if $x = x', y = y'$, and the interval $([u, u'], x, y) \subset E_\omega$. Now we define the space T_ω as the quotient space E_ω / \sim .

The space T_ω comes equipped with the map $p : \mathsf{T}_\omega \rightarrow \mathbb{R}^{|\omega|'} \times \mathbb{R}^{n-|\omega|'}$ induced by the obvious projection $(u, x, y) \rightarrow (x, y)$. Since X is compact, for each x , the polynomial $P(u, x)$ in (5.1) has finitely many intervals where it is negative, p is a ramified map with finite fibers.

For any fixed x , the u -polynomial in $P(u, x)$ in (5.1) can be viewed also as an element of the space $\mathcal{P}^{|\omega|} := \mathbb{R}_{\text{coef}}^{|\omega|}$, and as such belongs to a unique pure stratum $\mathsf{R}_{\omega'} := \mathsf{R}_{\omega'}(x) \subset \mathcal{P}^{|\omega|}$, where $\omega' \succeq \omega$ in $\Omega_{\langle |\omega| \rangle}$. Therefore, with the help of (5.1), each $x \in \mathbb{R}^{|\omega|'}$ has a well-defined combinatorial type $\omega(x) = \omega' \in \Omega_{\langle |\omega| \rangle}$ associated to it. As a result, $\mathbb{R}^{|\omega|'}$ is an $\Omega_{\langle |\omega| \rangle}$ -stratified space, and so is $E_\omega \subset \mathbb{R} \times \mathbb{R}^{|\omega|'} \times \mathbb{R}^{n-|\omega|'}$.

In fact, the space E_ω admit a "more intrinsic" stratification which is labeled by the elements of the poset $\Omega_{\langle d \rangle}^\bullet$, where $d = |\omega|'$. In a sense, this stratification is cruder than the $\Omega_{\langle |\omega| \rangle}$ -stratification of $\mathbb{R}^{|\omega|'}$. Here is the description of this $\Omega_{\langle |\omega|' \rangle}^\bullet$ -stratification.

For each point $(u_\star, x, y) \in E_\omega$, there is a unique closed interval $I_{u_\star, x} := [a, b]$ such that $u_\star \in [a, b]$, $P(u, x) \leq 0$ for all $u \in [a, b]$, and $I_{u_\star, x}$ is the maximal closed interval

possessing these two properties. Consider the real zero divisor $D_{(u_*, x)}$ of the u -polynomial $P(u, x)$ being restricted to the interval $I_{u_*, x}$. Its combinatorial type $\omega(u_*, x) \in \Omega^\bullet$ and is independent on the choice of u_* within the interval $I_{u_*, x}$. Thus, $\omega(u_*, x)$ depends only on the equivalence class of $(u_*, x, y) \in E_\omega$, a point in \mathbb{T}_ω .

On the other hand, if $\omega(x) = \omega(x')$ for some $x, x' \in \mathbb{R}^{|\omega|'}$, then there exist

$$(u_*, x, y), (u'_*, x', y) \in E_\omega$$

such that $\omega(u_*, x) = \omega(u'_*, x')$. Moreover, the orders which the intervals $I_{u_*, x}$ and $I_{u'_*, x'}$ occupy inside the sets $P^{-1}((-\infty, 0), x)$ and $P^{-1}((-\infty, 0), x')$, respectively, are the same. Stated differently, the combinatorial type $\omega(x) \in \Omega_{\langle |\omega| \rangle}$ determines the ordered sequence $\Xi(\omega(x))$ of types $\{\omega(u_*, x)\}_{u_*}$ for points in the fiber $p^{-1}(x)$ (cf. the discussion preceding Fig. 1).

Since the construction of the space \mathbb{T}_ω and its $\Omega_{\langle |\omega|' \rangle}^\bullet$ -stratification depends only on the combinatorial pattern $\omega \in \Omega_{\langle n \rangle}^\bullet$, we get:

Theorem 5.2. *For any transversally generic field v on a $(n+1)$ -manifold X and any v -trajectory γ with the tangency multiplicity pattern $\omega \in \Omega_{\langle n \rangle}^\bullet$, the $\Omega_{\langle |\omega|' \rangle}^\bullet$ -stratified topological type of the germ of the trajectory space $\mathcal{T}(v)$ at the point γ is determined by the combinatorial pattern ω alone.* \square

Example 5.1. For the transversally generic fields v on 4-folds X , there are 11 distinct local topological types for $\mathcal{T}(v)$. They are labeled by the elements of the poset from Fig. 2. \square

Next, we will employ similar considerations to describe the germ at γ of a cellular structure in \mathbb{T}_ω , a structure subordinate to the filtration of \mathbb{T}_ω by spaces which are labeled by the elements of the poset $\Omega_{\langle |\omega|' \rangle}^\bullet$. Eventually, these investigations will culminate in Theorem 5.3 below.

By choosing an appropriately narrow \hat{v} -adjusted neighborhood $U \subset \hat{X}$ of γ , for any x sufficiently close to the origin, the complex zeros of $P(u, x)$ from (5.1) can be separated into disjointed groups. These groups correspond to the real zeros $\{\alpha_i\}_i$ of the polynomial $P(u, 0)$ (which is a product of linear polynomials over \mathbb{R}) and reside in their vicinity. Moreover, each portion of the complex zero divisor of $P(u, x)$, taken within each group, by (5.1), is α_i -balanced.

Thus, for any \hat{v} -trajectory $\hat{\gamma} \subset U$, the “real” zero divisor $D_{\hat{\gamma}}$ splits into several “real” divisors $D_{\hat{\gamma}, i}$. Their combinatorial types are described by some elements $\hat{\omega}_i \in \Omega_{\langle \omega(i) \rangle}$ such that $|\hat{\omega}_i| \leq \omega(i)$ and $|\hat{\omega}_i| \equiv \omega(i) \pmod{2}$. In other words, $\omega_{\hat{\gamma}}$, the combinatorial type of $D_{\hat{\gamma}}$, is determined by an element of the product $\prod_i \Omega_{\langle \omega(i) \rangle}$. Evidently, there is a canonical map

$$\kappa : \prod_i \Omega_{\langle \omega(i) \rangle} \rightarrow \Omega_{\langle |\omega| \rangle}$$

that places $\{\hat{\omega}_i\}_i$ in a single array $\omega_{\hat{\gamma}} := (\hat{\omega}_1, \hat{\omega}_2, \dots)$.

For each i , we denote by x_i the ordered subset $\{x_{i,1}, x_{i,2}, \dots\}$ of the x -coordinates, amenable to the formula (5.1).

The decomposition of $P(u, x) = \prod_i P_i(u - \alpha_i, x_i)$ of the left-hand side of (5.1) into a product of monic depressed polynomials $P_i(u - \alpha_i, x_i)$ in $u - \alpha_i$ of degrees $j_i := \omega(i)$ gives rise a continuous map

$$(5.2) \quad A_\omega : \mathbb{R}^{|\omega|'} \rightarrow \prod_i \mathbb{R}_{\text{coef}, \alpha_i}^{\omega(i)-1}$$

from the space of x -coordinates $\mathbb{R}^{|\omega|'}$, to the product of polynomial spaces $\{\mathbb{R}_{\text{coef}, \alpha_i}^{\omega(i)-1}\}_i$ of equal dimension $|\omega|'$. If $x \neq x'$, then at least for one i , the vectors x_i, x'_i are distinct. Thus the $(u - \alpha_i)$ -polynomials $P_i(u - \alpha_i, x_i)$ and $P_i(u - \alpha_i, x'_i)$ must have distinct coefficients. As a result, A_ω is a 1-to-1 map. Evidently, by (5.1), A_ω is surjective. So A_ω establishes a smooth homeomorphism of the two spaces from (5.2).

With the help of Theorems 4.1 and 4.2, we get the cellular structures

$$\{\Phi_{\alpha_i, \hat{\omega}_i} : e_{\alpha_i, \hat{\omega}_i} \rightarrow R_{\alpha_i, \hat{\omega}_i}\}_{\hat{\omega}_i \in \Omega_{\langle \omega(i) \rangle}}$$

for each space $\mathbb{R}_{\text{coef}, \alpha_i}^{\omega(i)-1}$. By A_ω^{-1} from (5.2), this product of cellular structures in the target space of (5.2) gives rise to a cellular structure in the space $\mathbb{R}^{|\omega|'}$ of x -coordinates. Employing

$$\kappa : \prod_i \Omega_{\langle \omega(i) \rangle} \rightarrow \Omega_{\langle |\omega| \rangle}$$

, that structure

$$(5.3) \quad \Psi_{\hat{\omega}} := A_\omega^{-1} \circ \left(\prod_i \Phi_{\alpha_i, \hat{\omega}_i} \right) : \prod_i e_{\alpha_i, \hat{\omega}_i} \longrightarrow \mathbb{R}^{|\omega|'}$$

is consistent with the stratification of $\mathbb{R}^{|\omega|'}$, labeled by the elements $\kappa(\{\hat{\omega}_i\}_i)$ of the poset $\Omega_{\langle |\omega| \rangle}$.

In what follows, the cellular structure in the product $\mathbb{R}^{|\omega|'} \times \mathbb{R}^{n-|\omega|'}$ is chosen to be this cellular structure in $\mathbb{R}^{|\omega|'}$ given by (5.3), being multiplied by a single open cell $e^{n-|\omega|'} \approx \mathbb{R}^{n-|\omega|'}$.

Note that, so far, the cells in $\mathbb{R}^{|\omega|'}$ are labelled by the elements of the poset $\omega_{\preceq} \subset \Omega_{\langle |\omega| \rangle}$, and not by elements of the poset $\omega_{\bullet \preceq} \subset \Omega_{\langle |\omega|' \rangle}^\bullet$, appropriate for the trajectories in E_ω (see the discussion preceding Lemma 4.3).

Next, using the ramified map $p : T_\omega \rightarrow \mathbb{R}^{|\omega|'} \times \mathbb{R}^{n-|\omega|'}$ with finite fibers, we will employ the cellular structure (5.3) in the target space $\mathbb{R}^{|\omega|'}$ to produce a cellular structure in the source space T_ω , so that p will become a cellular map. That task will preoccupy us for a while...

With this goal in mind, we introduce *markers*, a new combinatorial contraption (see Fig. 1 and Fig. 4).

For each $\omega \in \Omega_{\langle d \rangle}$, consider the auxiliary u -polynomial

$$\wp_\omega(u) = \prod_{\{i \in \text{sup}(\omega)\}} (u - i)^{\omega(i)}$$

of degree that does not exceed d and shares the same parity with it.

For a given ω , we consider a pair (ω, k) , where the *marker* $k \in \text{sup}(\omega) \subset \mathbb{N}$ is such that:

- either $\omega(k) \equiv 0 \pmod{2}$ and $\wp_\omega(k - 0.5) > 0$, $\wp_\omega(k + 0.5) > 0$,
- or $\omega(k) \equiv 1 \pmod{2}$ and $\wp_\omega(k - 0.5) > 0$, $\wp_\omega(k + 0.5) < 0$.

We denote by $\Omega_{\langle d \rangle}^\mu$ the set of all marked pairs (ω, k) as above.

Let us denote by $\Upsilon(\omega) \subset \mathbb{N}$ the set of markers k associated with ω . Each marker $k \in \Upsilon(\omega)$, as an element of linearly ordered set $\Upsilon(\omega)$, acquires its ordinal p . We will denote the p -th marker in $\Upsilon(\omega)$ by k_p . Let $\Upsilon_p(\omega) \subset \mathbb{N}$ be the maximal set of consecutive natural numbers $j \geq k_p$ such that $\wp_\omega(u) \leq 0$ for all $u \in [k_p, j]$.

It is possible to extend the elementary operations M_j and I_j (merge and insert), introduced in (2.1)-(2.3) for the poset $\Omega_{\langle d \rangle}$, to the elements of the new set $\Omega_{\langle d \rangle}^\mu$.

The new merge operation $M_j^\mu : \Omega_{\langle d \rangle}^\mu \rightarrow \Omega_{\langle d \rangle}^\mu$ is shown in Fig. 4. Let

$$(5.4) \quad M_j^\mu(\omega, k) := (M_j(\omega), \mu_j(k))$$

, where $\mu_j(k) \in \Upsilon(M_j(\omega))$ is defined as follows.

If both j and $j + 1$ belong to the same set $\Upsilon_p(\omega)$, and the marker $k \in \Upsilon_q(\omega)$, $q \neq p$, then $\mu_j(k) = k$ as elements of the set $\Upsilon_q(M_j(\omega)) = \Upsilon_q(\omega)$. If j and $j + 1$ belong to $\Upsilon_p(\omega)$, and the marker $k \in \Upsilon_p(\omega)$, then again $\mu_j(k) = k$ as elements of the set $\Upsilon_p(M_j(\omega)) \subset \Upsilon_p(\omega)$.

At the same time, if $j \in \Upsilon_p(\omega)$, $j + 1 \in \Upsilon_{p+1}(\omega)$, and $k \in \Upsilon_{p+1}(\omega)$, then $\mu_j(k)$ is the unique minimal element in $\Upsilon_p(M_j(\omega))$. When $k \in \Upsilon_q(\omega)$ and $q \neq p + 1$, the marker keeps its minimal position within the subset $\Upsilon_q(\omega)$.

Similarly, the new insert operation

$$(5.5) \quad I_j^\mu(\omega, k) := (I_j(\omega), \mu_j(k))$$

is described as follows. Under I_j^μ , ω is subjected to the old insert operation I_j , while the marker $k \in \Upsilon_p(\omega)$ keeps its minimal position within the set $\Upsilon_p(\omega) \subseteq \Upsilon_q(I_j(\omega))$ (where q is uniquely determined by p and j), that is, $\mu_j(k) = k$ as the minimal elements of the appropriate sets.

Here is a slightly more geometrical look at the markers and their behavior, a look motivated by interactions of vector flows with the boundary $\partial_1 X$. Let $\hat{\gamma} \subset \hat{X}$ be a typical \hat{v} -trajectory in the vicinity of a given trajectory $\gamma \subset X$. The intersection $\hat{\gamma} \cap \partial_1 X = \{b_l\}_{1 \leq l \leq q}$, allows us to *shade* some of the intervals (b_l, b_{l+1}) in which $\partial_1 X$ divides $\hat{\gamma}$: the interval is shaded if it belongs to X . Using any auxiliary function $z : \hat{X} \rightarrow \mathbb{R}$ (as in Lemma 3.1 from [K2]), the shading is defined as the locus where $z|_\gamma \leq 0$.¹² Therefore, the ordered sequence of the multiplicities $\omega_{\hat{\gamma}} := (m(b_1), m(b_2), \dots, m(b_q))$ uniquely determines which intervals (b_l, b_{l+1}) along $\hat{\gamma}$ are shaded (cf. Fig. 1, where the shading corresponds to "strings" and "atoms"): each shaded interval can be marked with a unique lowest *odd*-multiplicity point in it; also each atom can be marked.

As in Section 2 (see the discussion preceding Fig. 1), we denote by $\text{sup}_{\text{odd}}(\omega)$ the points $l \in \mathbb{N}$ in the support of ω such that $\omega(l) \equiv 1 \pmod{2}$ and by $\text{sup}_{\text{ev}}(\omega)$ the points l in the

¹²Recall that z is chosen to possess the following properties: 1) 0 is a regular z -value and $z^{-1}(0) = \partial_1 X$, 2) $z^{-1}((-\infty, 0]) = X$.

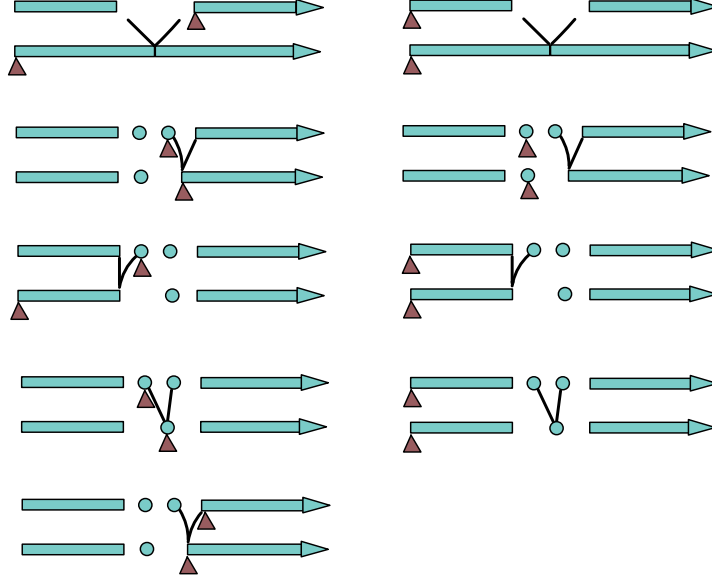


FIGURE 4. The rules by which the markers evolve under the merge operations (depicted as the V-shaped passages from top to bottom bars in each of the seven diagrams).

support of ω such that $\omega(l) \equiv 0 \pmod{2}$. We can count the elements of $\text{sup}_{\text{odd}}(\omega)$ and pick only the ones that acquire odd numerals in that count (in this way, we pick half of the elements in $\text{sup}_{\text{odd}}(\omega)$) (cf. Fig. 1). We denote this set by $\text{sup}_{\text{odd}}^+(\omega)$ and its complement by $\text{sup}_{\text{odd}}^-(\omega)$. We divide the points of $\text{sup}_{\text{ev}}(\omega)$ also into two complementary sets: the first one, $\text{sup}_{\text{ev}}^-(\omega)$, contains points that are bounded from below by a point from $\text{sup}_{\text{odd}}^+(\omega)$ and from above by a point from $\text{sup}_{\text{odd}}^-(\omega)$; the second one, $\text{sup}_{\text{ev}}^+(\omega)$, contains points that are bounded from below by a point from $\text{sup}_{\text{odd}}^-(\omega)$ and from above by a point from $\text{sup}_{\text{odd}}^+(\omega)$. In fact, $\Upsilon(\omega) = \text{sup}_{\text{odd}}^+(\omega) \cup \text{sup}_{\text{ev}}^+(\omega)$.

Consider $\omega' \succeq \omega$, where $\omega', \omega \in \Omega_{\langle m \rangle}$, and two elements-markers $k' \in \Upsilon(\omega')$ and $k \in \Upsilon(\omega)$. Our next task is to define a relation “ $k' \rightsquigarrow k$ ” between k' and k . Recall that ω can be obtained from ω' by a sequence of elementary merges and multiplicity 2 inserts as described in (2.1)-(2.3). Given a marker $k' \in \Upsilon(\omega')$, we will describe its evolution under these elementary transformations of ω' . An insertion of an even multiplicity point does not affect any marker below the insertion and shifts the location of the marker above the insertion by one. If two points from $\text{sup}_{\text{ev}}^+(\omega')$ merge and one of them is marked, the marker is placed at the location where the merge took place. If a marked point from $\text{sup}_{\text{ev}}^+(\omega')$ is merging with a point from $\text{sup}_{\text{odd}}^+(\omega')$, the location of the merge is marked. On the other hand, if a marked point from $\text{sup}_{\text{ev}}^+(\omega')$ is merging with a point from $\text{sup}_{\text{odd}}^-(\omega')$, then the marker is placed at the first point of $\text{sup}_{\text{odd}}^+(\omega')$ located below the merge—the

marker “travels down until it reaches a point of $\sup_{\text{odd}}^+(\omega')$ ”. Finally, if a marked point from $\sup_{\text{odd}}^+(\omega')$ merges with the adjacent point from $\sup_{\text{odd}}^-(\omega')$, the marker is placed at the first point of $\sup_{\text{odd}}^+(\omega')$ located below the merge. If one inserts a point of an even multiplicity, the marker does not change its location.

Therefore, when $\omega' \succeq \omega$, the marker k' defines a unique marker $k \in \Upsilon(\omega)$. In such a case, we say that k' *collapses* to k and write “ $k' \rightsquigarrow k$ ”.

There is a natural and already familiar map $\Xi : \Omega^\mu \rightarrow \Omega^\bullet$ which acts from the set of all marked pairs (ω, k) to the set Ω^\bullet of strings and atoms (see Fig. 1). By definition, Ξ takes each pair (ω, k) , $k \in \Upsilon(\omega)$, to the restriction of ω to the unique maximal interval $[k, j(k)]$ of indices in $\text{sup}(\omega)$ with the property $\wp_\omega(j) \leq 0$ for all $j \in [k, j(k)]$. Then, by a shift of indices, one reinterprets $\omega : [k, j(k)] \rightarrow \mathbb{N}$ as a map $\omega : [1, j(k) - k - 1] \rightarrow \mathbb{N}$, an element of Ω^\bullet . In fact, $\Xi : \Omega_{\langle 2n+2 \rangle}^\mu \rightarrow \Omega_{\langle n \rangle}^\bullet$ for each n .

In the following theorem, we combine the acquired knowledge about one particular cellular structure of the model spaces $\{\mathcal{T}_\omega\}$ with the local models for transversally generic vector fields (described in Lemma 1.2). This leads to a *local* purely combinatorial description of trajectory spaces for transversally generic fields. Regretfully, the formulation of the theorem is lengthy.

Theorem 5.3. *Let X be a compact smooth $(n+1)$ -manifold with boundary. Let v be a transversally generic vector field on X and γ its trajectory of the combinatorial type ω . Then the following statements hold:*

- *In the vicinity of γ , the trajectory space $\mathcal{T}(v)$ has a structure of the model $|\omega|'$ -dimensional finite cell complex \mathcal{T}_ω times $\mathbb{R}^{n-|\omega|'}$.*
- *Each open cell $E_{\hat{\omega}, k} \subset \mathcal{T}_\omega$ (of dimension $|\omega|' - |\hat{\omega}|'$) is indexed by an element $\hat{\omega} = \prod_i \hat{\omega}_i$ of the poset $\prod_i \Omega_{\langle \omega(i) \rangle}$ ¹³ together with a marker $k \in \Upsilon(\kappa(\hat{\omega}))$. Every point in $E_{\hat{\omega}, k}$ belongs to the pure stratum $\mathcal{T}(v, \Xi(\kappa(\hat{\omega}), k))$ of $\mathcal{T}(v)$, labeled by the element $\Xi(\kappa(\hat{\omega}), k) \in \Omega_{\langle n \rangle}^\bullet$.*
- *$E_{\hat{\omega}', k'} \subset E_{\hat{\omega}_\succeq, k}$ if and only if the following two conditions are satisfied:*
 - (1) *$\hat{\omega} \succeq \hat{\omega}'$, and*
 - (2) *the markers k and k' satisfy the relation “ $k \rightsquigarrow k'$ ”.*
- *Employing the attaching maps $\Psi_{\hat{\omega}} : e_{\hat{\omega}} := \prod_i e_{\hat{\omega}_i} \rightarrow \mathbb{R}^{|\omega|'}$ from (5.3), the space \mathcal{T}_ω can be assembled from the marked cells $\{(e_{\hat{\omega}}, k)\}_{\hat{\omega}, k}$ according to the rules described in Fig. 4. Each closed cell $E_{\hat{\omega}_\succeq, k}$ is the image of the cell $(e_{\hat{\omega}}, k)$ under the attaching map $\Psi_{\hat{\omega}, k} : \prod_{\hat{\omega}, k} (e_{\hat{\omega}}, k) \rightarrow \mathcal{T}_\omega$ based on (5.3).*
- *Each cell $E_{\hat{\omega}_\succeq, k}$, where $\kappa(\hat{\omega}) \succ \omega$, topologically is a positive cone with the apex at $\gamma \in \mathcal{T}(v)$ over a compact cell $S_{\hat{\omega}_\succeq, k}$. These cells $\{S_{\hat{\omega}_\succeq, k}\}$ form a link of the point γ in \mathcal{T}_ω . The rules that describe their incidence are similar to the incidence rules for $\{E_{\hat{\omega}_\succeq, k}\}$'s in \mathcal{T}_ω .*

¹³where the partial order in $\prod_i \Omega_{\langle \omega(i) \rangle}$ is the one of the posets product.

- \mathbb{T}_ω is equipped with a ramified cellular map $p : \mathbb{T}_\omega \rightarrow \prod_i \mathbb{R}_{\text{coef}, i}^{\omega(i)-1}$ whose fibers are of the cardinality $|\omega|/2$ at most. The image $E_{\hat{\omega}} := p(E_{\hat{\omega}, k})$ is a product of cells $\{E_{\hat{\omega}_i}\}_i$. The cellular structure $\{E_{\hat{\omega}_i}\}_{\hat{\omega}_i \in \Omega_{\langle \omega(i) \rangle}}$ in each space $\mathbb{R}_{\text{coef}, i}^{\omega(i)-1}$ is described in Theorem 4.1 and Lemma 4.3 in terms of the poset $(\Omega_{\langle \omega(i) \rangle}, \succ)$.

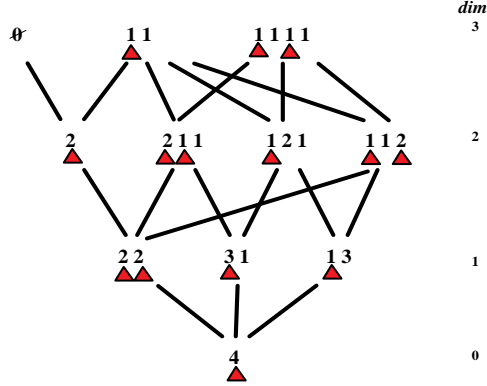


FIGURE 5. The Swallow Tail singularity poset $\Omega_{\langle 4 \rangle}$ with all possible markers k , denoted here by the symbol “ \triangle ”. The dimensions of the strata are shown on the right. The diagram indicates that the 3-complex $\mathbb{T}_{(4)}$ is assembled from three 3-cells, six 2-cells, four 1-cells and one 0-cell.

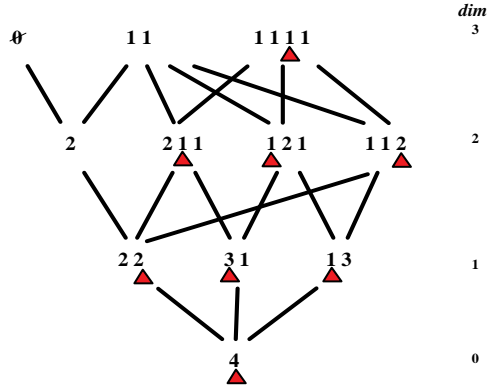


FIGURE 6. The poset $\Omega_{\langle 4 \rangle}$ with the downwards evolution “ \rightsquigarrow ” of a particular marker \triangle attached to $\omega = 1111$. The diagram indicates that the marked 3-cell $e_{1,1,1,\triangle,1}$ in $\mathbb{T}_{(4)}$ shares its boundary with three marked 2-cells: $e_{2,1,\triangle,1}$, $e_{1,\triangle,2,1}$, $e_{1,1,2,\triangle}$, and that each of these is attached to two marked 1-cells, and so on...

Proof. Recall that for $v \in \mathcal{V}^\dagger(X)$ and each trajectory $\hat{\gamma} \in \hat{X}$, the intersection $\hat{\gamma} \cap X$ consists of a number of closed ("shaded") intervals and few isolated points of even multiplicities. Each of these intervals or isolated points defines a singleton in the space $\mathcal{T}(v)$. We place a single marker " Δ " at one of the lower ends of the shaded intervals or at one of the isolated points, so that a trajectory $\hat{\gamma} \in \hat{X}$ and the marker $\Delta \in \hat{\gamma} \cap X$ will determine a point in $\mathcal{T}(v)$. Thus a pair $(\hat{\gamma}, \Delta)$, where $\hat{\gamma} \in \mathcal{T}(\hat{v})$, can be viewed a point of $\mathcal{T}(v)$. Here we distinguish between the combinatorial marker k and its geometrical realization $\Delta \in \partial_1 X \cap \hat{\gamma}$.

In the vicinity of a given trajectory $\gamma \subset X$, each trajectory $\hat{\gamma} \subset \hat{X}$ is determined by the intersection point $\hat{\gamma} \cap S$, where S denotes a local section of the \hat{v} -flow through a point on γ which resides slightly below the first intersection $a_1 \in \gamma \cap \partial_1 X$. By Theorems 4.1 and 4.2, the combinatorial types $\hat{\omega} \in \prod_i \Omega_{\langle \omega(i) \rangle}$, or rather their κ -images, label cells $E_{\hat{\omega}}$ of codimension $|\kappa(\hat{\omega})|'$ in S . To define the corresponding cell decomposition of $\mathcal{T}(v)$ in the vicinity of γ , we need multiple copies $\{E_{\hat{\omega},k}\}_k$ of each cell $E_{\hat{\omega}}$, the copies that are indexed by all admissible markers k . Their number is the cardinality of the set $\Upsilon(\kappa(\hat{\omega}))$. For each $\hat{\gamma}$, there is a canonical correspondence between the markers $k \in \Upsilon(\kappa(\hat{\omega}))$ and the connected components $\hat{\tau} \in \pi_0(\hat{\gamma} \cap X)$. So we will use elements $k \in \Upsilon(\kappa(\hat{\omega}))$ and $\hat{\omega} \in \prod_i \Omega_{\langle \omega(i) \rangle}$ to label all the cells in $\mathcal{T}(v)$.

We are going to show that, for any $\hat{\omega} \succ \hat{\omega}'$ in $\prod_i \Omega_{\langle \omega(i) \rangle}$, a k -marked cell $E_{\hat{\omega},k}$ is incident to a k' -marked cell $E_{\hat{\omega}',k'}$, where $k \rightsquigarrow k'$, if and only if $E_{\hat{\omega}}$ is incident to $E_{\hat{\omega}'}$ in S .

In the vicinity of γ , let us employ the special coordinates (u, x, y) as in Lemma 1.2. Consider a smooth path $\{w(t)\}_{0 \leq t < 1}$ in the pure stratum $S(\hat{\omega}) \subset S$ so that $\lim_{t \rightarrow 1} w(t) = w_1 \in S(\hat{\omega}')$. The corresponding trajectories $\gamma_{w(t)} \subset \hat{X}$ converge to a trajectory γ_{w_1} in such a way that the sets $\gamma_{w(t)} \cap \partial_1 X$ go through merges of adjacent points or/and insertions of even multiplicity points. Examining the evolution of each element $\hat{\tau}_t \in \pi_0(\gamma_{w(t)} \cap X)$ into an element $\hat{\tau}' \in \pi_0(\gamma_{w_1} \cap X)$, we see that it is captured by a marker collapse $\Delta_t \rightsquigarrow \Delta'$ as depicted in Fig. 4. Also glance at Fig. 6 which gives an example of such evolution.

With the rigid rules for the collapses $\tau \rightsquigarrow \tau'$ from Fig. 4 in place, we can reconstruct the germ \mathcal{T}_γ of $\mathcal{T}(v)$ at γ from the cell complex S (whose cells are indexed by the elements of the poset $(\prod_i \Omega_{\langle \omega(i) \rangle}, \succ)$), and ultimately, from $\omega = \omega(\gamma)$ itself. Speaking informally, in order to attach a cell $E_{\omega_\succeq, k}$ to a cell $E_{\omega', k'}$, first we verify whether $k \rightsquigarrow k'$, and if the verification leads to a positive answer, we use the same map that attaches E_{ω_\succeq} to $E_{\omega'}$ in S .

Let us describe a more formal construction of $\mathcal{T}_\gamma \approx \mathsf{T}_\omega \times \mathbb{R}^{n-|\omega|'}$.

For $\hat{\omega} = \{\hat{\omega}_i \in \Omega_{\langle \omega(i) \rangle}\}_i$, let $e_{\hat{\omega}} = \prod_i e_{\hat{\omega}_i}$, where the hypersurface

$$(5.6) \quad e_{\hat{\omega}_i} \subset \text{Sym}^{|\sup(\hat{\omega}_i)|}(\mathbb{R}) \times \text{Sym}^{\frac{\omega(i)-|\hat{\omega}_i|}{2}}(\mathbb{H})$$

consists of divisors (D'_i, D''_i) such that $D'_i + D''_i + \tau(D''_i)$ is i -balanced.

Then T_ω is the quotient of the space

$$(5.7) \quad \mathsf{Z}_\omega := \coprod_{\hat{\omega} \in \prod_i \Omega_{\langle \omega(i) \rangle}, k \in \Upsilon(\kappa(\hat{\omega}))} (e_{\hat{\omega}}, k)$$

by the equivalence relation " $(z, k) \sim (z', k')$ " that is defined as follows:

- $\hat{\omega} \succ \hat{\omega}'$ in the poset $\prod_i (\Omega_{\langle \omega(i) \rangle}, \succ)$,
- $z \in \partial e_{\hat{\omega}}$, $z' \in e_{\hat{\omega}'}^\circ$ are such that $\Psi_{\hat{\omega}}(z) = \Psi_{\hat{\omega}'}(z')$ in the space $\prod_i \mathbb{R}_{\text{coef}, i}^{\omega(i)-1}$, where the Ψ -maps are defined by (5.3),
- $k \rightsquigarrow k'$.

Each closed cell $E_{\hat{\omega}, k} \subset T_\omega$ is defined as the equivalence class of the set $(e_{\hat{\omega}}, k)$ in the quotient space Z_ω / \sim . Its interior is homeomorphic to an open ball $e_{\hat{\omega}}^\circ$ of dimension $|\omega|' - |\hat{\omega}|'$.

As a result, for a transversally generic v -flow, we have precise instructions for assembling the germ of a cell complex \mathcal{T}_γ associated with each trajectory $\gamma \in \mathcal{T}(v)$, or rather, with its combinatorial type ω .

In the vicinity of γ , the projection $p : \mathcal{T}(v) \rightarrow S$ is evidently a finitely-ramified cellular map with respect to the cellular structures $\{E_{\hat{\omega}, k}\}_{\hat{\omega}, k}$ and $\{E_{\hat{\omega}}\}_\omega$.

Note that not every cell $E_{\hat{\omega}} \subset S$ is the p -image of some cell from $\mathcal{T}(v)$: the cells in S that are pierced by the trajectories $\hat{\gamma} \subset \hat{X}$ with the property $\hat{\gamma} \cap X = \emptyset$ are not in $p(\mathcal{T}(v))$. However, each $E_{\hat{\omega}}$ with $\text{sup}(\hat{\omega}) \neq \emptyset$ belongs to $p(\mathcal{T}(v))$.

For each $\hat{\omega}$, the p -fiber over the pure stratum $S(\kappa(\hat{\omega}))^\circ \subset S$ is of the cardinality $\#[Y(\kappa(\hat{\omega}))]$. We would like to get an upper bound for $\#[Y(\kappa(\hat{\omega}))]$ in terms of the ω . In fact, $m(\gamma)/2 = |\omega|/2$ is such an upper bound. Indeed, the divisor D_γ can be resolved in a divisor with $m(\gamma)$ simple roots. Recall that $m(\gamma)$ must be even, and, by Theorem 3.3 and Theorem 3.5 from [K2], any potential resolution of D_γ (with the combinatorics described by elements of $\kappa(\prod_i \Omega_{\langle \omega(i) \rangle})$) is realized in the vicinity of γ . Therefore $m(\gamma)/2$ shaded intervals do occur in the vicinity of γ . In fact, for transversally generic v , the number $m(\gamma)/2 \leq \dim(X)$ is the maximal possible cardinality of the p -fibers in the vicinity of γ .

Next we turn to describing the positive cone structure of $\mathcal{T}(v)$ at γ , which is consistent with its cellular structure $\{E_{\hat{\omega}, k}\}_{\hat{\omega}, k}$. In order to construct a base $\sigma_{\hat{\omega}}$ for the cone structure in each cell $e_{\hat{\omega}} = \prod_i e_{\hat{\omega}_i}$ (see (4.5) and (5.3)), consider a set of disjoint closed 2-balls $\{B(i) \subset \mathbb{C}\}_i$ of radius $1/3$ and with the center at $(i, 0) \in \mathbb{C}$. We denote by $\sigma_{\hat{\omega}} \subset e_{\hat{\omega}}$ the set of complex conjugation-invariant divisor pairs $(D', D'') = \oplus_i (D'_i, D''_i)$, where the divisor $D'_i + D''_i + \tau(D''_i)$ of degree $\omega(i)$ is i -balanced and the $\text{sup}(D'_i + D''_i) \subset B(i)$; furthermore, at least for one i , $\text{sup}(D'_i + D''_i)$ is not contained in the interior of $B(i)$. The group \mathbb{R}_+^* acts semi-freely on $e_{\hat{\omega}} = \prod_i e_{\hat{\omega}_i}$ by the diagonal action that applies a t -dilatation centered on the point $(i, 0) \in \mathbb{C}$ to each pair $(D'_i, D''_i) \in e_{\hat{\omega}_i}$. Evidently, for each pair (D', D'') there is a single $t \in \mathbb{R}_+^*$ so that $t(D', D'') \in \sigma_{\hat{\omega}}$. This \mathbb{R}_+^* -action extends to the space Z_ω in (5.3), the action on the markers being trivial.

By Theorem 4.1 and with the help of the Viète map V , a similar semi-free action is available on the space $\prod_i \mathbb{R}_{\text{coef}, i}^{\omega(i)-1}$. Since the maps $\prod_i \Phi_{i, \hat{\omega}_i}$ are \mathbb{R}_+^* -equivariant, the quotient space $T_\omega = Z_\omega / \sim$ inherits a semi-free \mathbb{R}_+^* -action for which γ is the only fixed point.

Let $S_{\hat{\omega}, k}$ be the image of $\sigma_{\hat{\omega}} \times k$ under the obvious map $Z_\omega \rightarrow T_\omega$. Any nontrivial \mathbb{R}_+^* -trajectory meets $S_{\hat{\omega}, k}$ at a singleton. Therefore $E_{\hat{\omega}, k}$ is a positive cone over $S_{\hat{\omega}, k}$. By their constructions, these cone structures in the individual cells $E_{\hat{\omega}, k}$ are well-correlated and

produce a positive cone structure in T_ω . As a result, $\{S_{\tilde{\omega},k}\}_{\tilde{\omega},k}$ define a cellular structure in the link of γ in $\mathcal{T}_\gamma = T_\omega \times \mathbb{R}^{n-|\omega|'}$. \square

Here is a short summary of what we have established in this paper: a transversally generic v -flow generates a stratification $\{\mathcal{T}(v, \tilde{\omega})\}_{\tilde{\omega} \in \Omega_{\bullet', \langle n \rangle}}$ of the trajectory space $\mathcal{T}(v)$, which is consistent (in a subtle way!) with, but cruder than, the γ -local cellular structure

$$\{E_{\tilde{\omega},k} \times \mathbb{R}^{n-|\omega|'}\}_{\tilde{\omega} \in \prod_i \Omega_{\langle \omega(i) \rangle}, k \in T(\kappa(\omega))}$$

in the model space $\mathcal{T}_\gamma \approx T_\omega \times \mathbb{R}^{n-|\omega|'}$ that we just have described. Each cell $E_{\tilde{\omega},k} \times \mathbb{R}^{n-|\omega|'}$ belongs to the stratum $\mathcal{T}(v, \tilde{\omega})$, where $\tilde{\omega} := \Xi(\kappa(\tilde{\omega})) \in \Omega_{\bullet', \langle n \rangle}$. In the process, distinct cells could acquire the same label $\tilde{\omega} \in \Omega_{\bullet', \langle n \rangle}$.

This understanding of the cellular structure of the spaces of real polynomials of the degree $2n + 2$ —the structure which reflect the universal posets $\Omega_{\langle 2n+2 \rangle}$ and $\Omega_{\bullet', \langle n \rangle}$ —will enable us, in the papers to follow, to define new rich characteristic classes of transversally generic flows on $(n + 1)$ -manifolds with boundary.

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